

On the Distribution of Zeroes of Artin-Schreier L-functions

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Abstract

We study the distribution of the zeroes of the L-functions of curves in the Artin-Schreier family. We consider the number of zeroes in short intervals and obtain partial results which agree with a random unitary matrix model.

1 Introduction and statement of main results

Let p be a prime number, $q = p^n$ its power. Let d be a natural number prime to p . We consider the family of curves over \mathbf{F}_q defined by an equation of the form

$$y^p - y = f(x) = a_d x^d + \dots + a_1 x \quad (1)$$

with $a_i \in \mathbf{F}_q$, $a_d \neq 0$ and $a_k = 0$ for all k divisible by p (every curve defined by an equation of the form $y^p - y = f(x)$ with $f \in \mathbf{F}_q[x]$ of degree d is a twist of a curve of the form (1) satisfying this condition, see section 3.2). We call such curves Artin-Schreier curves, or A-S curves for short, and the corresponding family the A-S family (with parameter d).

Denote by Ψ the set of nontrivial additive characters of \mathbf{F}_p . It is known that the L-function of the normalisation of the projective closure of a curve defined by (1) factors into primitive L-functions as follows:

$$L_f(z) = \prod_{\psi \in \Psi} L_{f,\psi}(z) \quad (2)$$

with

$$L_{f,\psi}(z) = \exp \left(\sum_{r=1}^{\infty} \sum_{\alpha \in \mathbf{F}_{q^r}} \psi \left(\text{tr}_{\mathbf{F}_{q^r}/\mathbf{F}_p} f(\alpha) \right) \frac{z^r}{r} \right). \quad (3)$$

Each of the $p-1$ factors in (2) is a polynomial of degree $d-1$ with all zeroes having absolute value $q^{-1/2}$ due to the Riemann Hypothesis for curves over finite fields (see section 3.3).

Denote by \mathcal{F}_d the set of polynomials f of the form in (1) satisfying the stated conditions. We denote

$$T_{f,\psi}^r = \sum_{i=1}^{d-1} \rho_i^r,$$

where ρ_i are the normalised zeroes of $L_{f,\psi}$ counting multiplicity. The zeroes are normalised as follows: $\rho_i = q^{1/2} \lambda_i^{-1}$, where λ_i are the zeroes of $L_{f,\psi}$. We have $|\rho_i| = 1$. Note that the normalised zeroes are proportional to the inverse zeroes of $L_{f,\psi}(z)$ and we preserve this normalisation convention for zeroes of L-functions throughout the paper. The quantity $T_{f,\psi}^r$ is the trace of the r -th power of the Frobenius element corresponding to the L-function $L_{f,\psi}$. For any finite set A and a function $X : A \rightarrow \mathbf{C}$ (we denote by $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$ the set of integers, rational, real and complex numbers respectively) we denote by $\langle X(a) \rangle_{a \in A}$ the average of $X(a)$ as a runs uniformly through A , in other words

$$\langle X(a) \rangle_{a \in A} = \#A^{-1} \sum_{a \in A} X(a).$$

Denote

$$M_d^r = \langle T_{f,\psi}^r \rangle_{f \in \mathcal{F}_d}.$$

It does not depend on the choice of $\psi \in \Psi$, see section 3.4. Our main result is the following

Theorem 1.

$$M_d^r = -e_{p,r} q^{r/p-r/2} + O\left(r q^{r/2-(1-1/p)d} + q^{-r/2}\right),$$

where

$$e_{p,r} = \begin{cases} 0, & (r,p) = 1, \\ 1, & p|r \end{cases} \quad (4)$$

(the implicit constant is absolute, i.e. does not depend on p, q, r, d).

Note that the error term in Theorem 1 is small when $r \leq (2 - 2/p - \epsilon)d$ for any fixed $\epsilon > 0$. For $p > 2$ Theorem 1 suggests that for $d \rightarrow \infty$ the zeroes of all the L-functions in this family when taken together are distributed quite uniformly on the unit circle. We conjecture that the average number of zeroes of $L_{f,\psi}$ contained in an arc of length $O(1/d)$ on the unit circle as f varies uniformly through \mathcal{F}_d tends to the length of the arc divided by 2π as $d \rightarrow \infty$. For arcs of length l s.t. $ld \rightarrow \infty$ this has recently been proved in [3]. The conjecture is related to the random unitary matrix model for A-S L-functions which we present in section 4. We are only able to obtain a weaker result with the arc replaced by a window function with bounded frequency. Denote by $\mathcal{S}(\mathbf{R})$ the space of smooth complex-valued functions on the real line with all derivatives decaying faster than any power of t at infinity (the Schwartz space).

Theorem 2. Assume $p > 2$. Let $V \in \mathcal{S}(\mathbf{R})$ be a function s.t. its Fourier transform

$$\hat{V}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(t) e^{-ist} dt$$

is supported on the interval $(-2 + 2/p, 2 - 2/p)$. Denote

$$v_d(t) = \sum_{n=-\infty}^{\infty} V(d(t + 2\pi n)).$$

This function has period 2π . Let θ be any real number. Denote

$$S_f = \sum_{j=1}^{d-1} v_d(\theta_j - \theta),$$

where $\rho_j = e^{i\theta_j}$ are the normalised zeroes of $L_{f,\psi}$, θ_j being real numbers well defined modulo 2π . Then

$$\langle S_f \rangle_{f \in \mathcal{F}_d} = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(t) dt + o(1)$$

as $d \rightarrow \infty$ and q may vary as we please, i.e. there is a bound on the decay rate of the $o(1)$ term which depends on d and V but not on q (in fact the error becomes smaller as q grows).

We conjecture that this in fact holds for any $V \in \mathcal{S}(\mathbf{R})$, but with the presently existing methods it seems difficult to prove.

We also consider some nonlinear statistics of the zeroes of L-functions in the A-S family. Let $V \in \mathcal{S}(\mathbf{R}^2)$ be a two variable window function and $v_{d-1}(t, u)$ the periodic window function associated with V by

$$v_{d-1}(t, u) = \sum_{m, n=-\infty}^{\infty} V((d-1)(t + 2\pi m), (d-1)(u + 2\pi n)).$$

Let $\rho_1, \dots, \rho_{d-1}$ be the normalised zeroes of $L_{f,\psi}$ for $f \in \mathcal{F}_d$, $\rho_j = e^{i\theta_j}$ and let θ be some fixed real number. We consider the 2-level density function (at θ):

$$S_{\theta}^2(f, \psi) = \sum_{\substack{1 \leq j, k \leq N \\ j \neq k}} v_{d-1}(\theta_j - \theta, \theta_k - \theta).$$

For a function $V \in \mathcal{S}(\mathbf{R}^2)$ we define its Fourier transform by

$$\hat{V}(\eta, \xi) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(t, u) e^{-i\eta t - i\xi u} dt du.$$

In section 6 we will prove the following

Theorem 3. Assume $p > 2$. Let $V \in \mathcal{S}(\mathbf{R}^2)$ be a window function s.t. its Fourier transform $\hat{V}(\eta, \xi)$ is supported on the set $|\eta| + |\xi| \leq 1$. Then for all θ ,

$$\langle S_\theta^2(f, \psi) \rangle_{f \in \mathcal{F}_d} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(t, u) \left(\frac{\sin((t-u)/2)}{(t-u)/2} \right)^2 dt du + o(1)$$

as $d \rightarrow \infty$.

The connection with the random unitary matrix model is discussed in section 6.

We also consider the subfamily of A-S curves defined by (1) with odd polynomial $f(x)$. For an odd natural number d denote by \mathcal{O}_d the subset of $f \in \mathcal{F}_d$ s.t. $f(x) = -f(-x)$. In section 8 we prove the following

Theorem 4. Assume $p > 2$ and d is odd. There exists a positive constant C (in fact any $C < 1$ will do) s.t.

$$\langle T_{f, \psi}^r \rangle_{f \in \mathcal{O}_d} = -e_{2,r} + O\left(rq^{-r/6}\right),$$

provided that $r < Cp \log_q d$ and $r < d/4$ ($e_{2,r}$ is defined by (4)).

This result agrees with a random symplectic matrix model for the L-zeroes in the family \mathcal{O}_d (see section 8).

We consider a more general type of families of L-functions corresponding to Dirichlet characters and show that the A-S family (as well as the A-S family with odd polynomials) is a special case. We also indicate how our results for the A-S family can be generalised to such families. This occupies section 7.

We also consider a related problem - the distribution of the number of points of a curve varying uniformly in a family of A-S curves. The proofs of our results are presented in section 9. This part is independent of the rest of our work and the interested reader may skip to section 9 after section 3. We consider the distribution of the number of points on the curve C_f as f varies uniformly through the A-S family and $d \rightarrow \infty$. Here we denote by C_f the normalisation of the projective closure of the curve defined by (1) for $f \in \mathbf{F}_q[x]$. We define \mathcal{G}_d to be the set of all monic degree d polynomials in $\mathbf{F}_q[x]$. For the problem of the distribution of the number of points it is more convenient to consider the family of A-S curves defined by the polynomials in \mathcal{G}_d . It is not difficult to adjust the statements and proofs for the case of the family \mathcal{F}_d .

For the rest of this section let r be a fixed natural number. We will see in section 3.2 that the number of \mathbf{F}_{q^r} -rational points on an A-S curve C_f always equals 1 modulo p and so we denote $N(f) = (\#C_f(\mathbf{F}_{q^r}) - 1)/p$. Our results concern the distribution of $N(f)$ as f varies uniformly in the family \mathcal{G}_d .

We denote by $\pi(e)$ the number of monic degree e irreducible polynomials in $\mathbf{F}_q[x]$. We denote by $B(t)$ the Bernoulli random variable which assumes 1 with probability t and 0 with probability $1 - t$. For two random variables X, Y we write $X \sim Y$ if they have the same distribution.

Theorem 5. Let p, n be fixed. For each $e|r$ let $X_{e,1}, \dots, X_{e,\pi(e)} \sim B(1/p)$ be random variables with all $\{X_{e,i}\}_{e|r, 1 \leq i \leq \pi(e)}$ independent. Then for $d \geq q^r$ the following holds:

(i) If $(r, p) = 1$, the distribution of $N(f)$ equals that of

$$\sum_{e|r} e \sum_{i=1}^{\pi(e)} X_{e,i}.$$

In particular the mean value of $N(f)$ is q^r/p .

(ii) If $p|r$ then the distribution of $N(f) - q^{r/p}$ equals that of

$$\sum_{\substack{e|r \\ (r/e, p)=1}} e \sum_{i=1}^{\pi(e)} X_{e,i}.$$

In particular the mean value of $N(f)$ is

$$\frac{q^r}{p} + \left(1 - \frac{1}{p}\right) q^{r/p}.$$

We also consider what happens when p, n are allowed to vary.

Theorem 6. Let p, d both tend to infinity and $n, r = 1$. Then $N(f)$ converges in distribution to the Poissonian distribution with mean 1, i.e. $\mathbf{P}(m) = e^{-m}/m!$

Theorem 7. Let p be fixed and $n, d \rightarrow \infty$.

(i) If $(r, p) = 1$ or $p > 2$ then

$$p^{1/2} ((1 - 1/p)r)^{-1/2} q^{-r/2} \left(N(f) - \frac{q^r}{p} \right)$$

converges in distribution to the Gaussian distribution with mean 0 and variance 1.

(ii) If $p = 2$ and r is even then

$$2r^{-1/2} q^{-r/2} \left(N(f) - \frac{q^r}{p} - \frac{2q^{r/2}}{r} \right)$$

converges in distribution to the Gaussian distribution with mean 0 and variance 1.

Theorem 8. Let p, d both tend to infinity and assume that $n > 1$ (not necessarily constant) or $r > 1$. Then

$$p^{1/2} r^{-1/2} q^{-r/2} \left(N(f) - \frac{q^r}{p} \right)$$

converges in distribution to the Gaussian distribution with mean 0 and variance 1.

The paper is organised as follows: in the next section we review related work dealing with similar problems for other families of L-functions. In section 3 we provide the necessary background on A-S curves and L-functions. In section 4 we describe the random unitary matrix model for the A-S family of L-functions. In section 5 we prove Theorem 1 and derive Theorem 2 from it. In section 6 we consider the 2-level density statistics of the L-zeroes in the A-S family and obtain results which agree with the random unitary matrix model. In section 7 we reformulate our main results in terms of Dirichlet L-functions over $\mathbf{F}_q[x]$ and generalise them to suitable families of Dirichlet L-functions. In section 8 we consider the family of A-S curves defined by (1) with f odd, for which we formulate conjectures corresponding to a random symplectic matrix model and provide evidence for them in the form of theorem 4. In section 5.3 we will discuss the situation with $p = 2$. In section 9 we prove our results on the distribution of the number of points on curves in the A-S family.

2 Related work

2.1 The hyperelliptic ensemble

The main inspiration for the present work is the paper [16], which studies similar questions and obtains similar results for an ensemble of hyperelliptic curves. We briefly present the content of that work. One considers the family of curves over \mathbf{F}_q , with q odd, defined by equations of the form $y^2 = f(x)$, with f monic squarefree of degree d , with d odd. Denote by \mathcal{H}_d the set of all such polynomials f . For $f \in \mathcal{H}_d$ the curve defined by $y^2 = f(x)$ has an L-function which is a polynomial with integer coefficients, which can be written as

$$L_f(z) = \prod_{i=1}^{d-1} (1 - \rho_i q^{1/2} z),$$

where ρ_i are the normalised zeroes of $L_f(z)$ satisfying $|\rho_i| = 1$.

Denote $T_f^r = \sum_{i=1}^{d-1} \rho_i^r$. It is shown in [16] that

$$\langle T_f^r \rangle_{f \in \mathcal{H}_d} = \begin{cases} -e_{2,r} + E_{d,r}, & 0 < r < d, \\ E_{d,r}, & r \geq d, \end{cases}$$

where $e_{2,r}$ is given by (4) and the error term $E_{d,r}$ satisfies

$$E_{d,r} = O_q \left(dq^{r/2-d} + dq^{-d/2} \right)$$

for $r \neq d-1$ (if $r = d-1$ an additional summand of $-q/(q-1)$ appears). This provides evidence in favor of the random symplectic matrix model for the L-functions of hyperelliptic curves because

$$\langle \text{tr} U^r \rangle_{U \in \mathbf{USp}(d-1)} = \begin{cases} -e_{2,r}, & r < d, \\ 0, & r \geq d. \end{cases}$$

This result is used to obtain a result about the average number of zeroes in short intervals, which again agrees with the random symplectic matrix model. Namely, let V be as in Theorem 2 but with Fourier transform supported in $(-2, 2)$ and define v_d as in Theorem 2. Denote $Z_f = \sum_{j=1}^{d-1} v_d(\theta_{f,j})$, where $\rho_{f,j} = e^{i\theta_{f,j}}$ are the normalised zeroes of L_f ($\theta_{f,j}$ are real numbers well defined modulo 2π) and $Z_U = \sum_{j=1}^{d-1} v_d(\theta_{U,j})$, where $\rho_{U,i} = e^{i\theta_{U,i}}$ are the eigenvalues of a matrix $U \in \mathbf{USp}(d-1)$. Then

$$\langle Z_f \rangle_{f \in \mathcal{H}_d} = \langle Z_U \rangle_{U \in \mathbf{USp}(d-1)} + o(1)$$

as $d \rightarrow \infty$ and q is fixed.

Note that \mathcal{H}_d has on the order of q^d elements while $\#\mathcal{F}_d = O(q^{d-d/p})$, which means that the A-S family is sparser and we get less averaging. As a result we get large errors in our estimate for M_r^d already for $r > d(2 - 2/p)$, while in the hyperelliptic case it is possible to obtain small errors for $r < 2d$.

2.2 Constant d

Much more is known about the statistics of zeroes for various families of L-functions over finite fields if the degree of the family is held constant while $q \rightarrow \infty$. For an L-function of the form $L(z) = \prod_{i=1}^m (1 - q^{1/2} \rho_i z)$, $|\rho_i| = 1$ we attach the class of unitary matrices with eigenvalues ρ_1, \dots, ρ_m . For many families of L-functions (e.g. the hyperelliptic family, the family of Dirichlet characters and families similar to our A-S family) it was shown by Katz and Sarnak (see [8]) that as $q \rightarrow \infty$ and m is fixed the classes corresponding to the L-functions of the objects in the family become equidistributed in the set of conjugacy classes of a suitable compact group of matrices (endowed with the measure induced by the Haar measure on the group), usually $\mathbf{U}(m)$, $\mathbf{USp}(m)$, the orthogonal group or some similar group, called the symmetry type of the family. These results cannot be extended to the case $m \rightarrow \infty$ (there is no meaning to equidistribution in a varying space) but the symmetry types observed for constant m can be used to give random matrix models to families of L-functions. The model for our family of A-S L-functions is presented in section 4.

2.3 The number of points on curves

The distribution of the number of points on curves in various families has been studied extensively in recent years. It follows from (5), (6) below that the number of points on a curve C with normalised L-zeroes ρ_1, \dots, ρ_{2g} (g is the genus of the curve) is $q + 1 - q^{1/2} \sum_{i=1}^{2g} \rho_i$. The distribution of $T_C^1 = \sum_{i=1}^{2g} \rho_i$ as C varies through some family of curves over \mathbf{F}_q with genus g as $g \rightarrow \infty$ is considered in [9] for the hyperelliptic family, in [1], [18] for the family of trigonal covers of \mathbf{P}^1 (with further generalisation to l -gonal covers, l a prime dividing $q-1$) and in [2] for the family of plane curves. A family of curves in higher dimensional projective spaces has been studied in [10]. The distribution of the number of points for the family of A-S curves is considered in section 9.

2.4 Number fields

Results similar to Theorem 2 have previously been obtained for families of L-functions over number fields. The family of quadratic Dirichlet L-functions (with varying modulus) is considered in [11] and the family of all Dirichlet L-functions with given modulus is considered in [6].

For example let q be a prime number and χ a Dirichlet character modulo q . Let $L(s, \chi)$ be the corresponding L-series and ρ_1, ρ_2, \dots its sequence of non-trivial zeroes ordered by increasing absolute value. For simplicity we assume the Generalised Riemann Hypothesis (although it is not assumed in [6]) so that $\text{Re} \rho_i = 1/2$. Denote $\gamma_i = \text{Im} \rho_i$. Denote by Ξ the set of nontrivial Dirichlet characters modulo q . The average number of L-zeroes satisfying $|\gamma_i| < T$ as χ varies uniformly in Ξ , T is fixed and $q \rightarrow \infty$ is known to be

$$\langle N(T, \chi) \rangle_{\chi \in \Xi} \sim \frac{T}{\pi} \log qT,$$

so we normalise $\delta_i = \frac{\log q}{\pi} \gamma_i$ (now we expect on average one zero with $|\delta_i| < 1$). Let $V \in \mathcal{S}(\mathbf{R})$ be a window function, $Z_\chi = \sum_{i=1}^{\infty} V(\delta_i)$. It is shown in [6] that if \hat{V} is supported on $[-2, 2]$ then

$$\langle Z_\chi \rangle_{\chi \in \Xi} \rightarrow \int_{-\infty}^{\infty} V(t) dt$$

as $q \rightarrow \infty$. Studying nonlinear statistics they obtain agreement with a random unitary matrix model (with matrix size around $\log q$) for restricted classes of window functions.

3 Background on Artin-Schreier curves and L-functions

The material reviewed in this section can be found in [13],[15],[17].

3.1 Notation and conventions

The notation and conventions introduced in this subsection apply to the entire paper, including the introduction.

When we use the O -notation (asymptotic bound) the implicit constant is absolute, except when the bounded quantity depends on a window function V , in which case it may depend on V . If there are additional parameters upon which the bound depends we write them explicitly as a subscript (e.g. $f = O_\epsilon(g)$). The o -notation is always used for $d \rightarrow \infty$ and $f = o(g)$ implies that $f/g \rightarrow 0$ as $d \rightarrow \infty$ regardless of how the other parameters on which f, g depend vary, except possibly for a window function V which is always assumed to be fixed.

For a pair of integers m, n we denote by (m, n) their greatest common divisor. For a pair of polynomials f, g , (f, g) denotes their greatest monic common divisor and $f \bmod g$ denotes the residue class of f modulo g .

For a finite set S we denote by $\#S$ its number of elements.

3.2 Geometric properties of Artin-Schreier curves

Let F be a field of characteristic $p > 0$. An Artin-Schreier (A-S shortly) curve over F is the normalisation of the projective closure of the affine curve defined by an equation of the form $y^p - y = f(x)$ with $f \in F[x]$ a polynomial of degree $d > 0$. We denote this curve by C_f . If $(d, p) = 1$ then C_f is geometrically irreducible [17, §1.4.2] (however this condition is far from necessary for geometric irreducibility). We assume throughout that $(d, p) = 1$. The affine part of C_f is smooth, as $\partial(y^p - y - f(x))/\partial y = -1$ never vanishes. The curve C_f has exactly one point (always F -rational) outside its affine part and the genus of C_f is $g = (p-1)(d-1)/2$ (this follows from the material of [13, §4.6.2]).

Let (x, y) be an affine point on the curve C_f , possibly defined over the algebraic closure of F . Then $(x, y), (x, y+1), \dots, (x, y+p-1)$ are all points on C_f and these are all the points of C_f with abscissa x . We see in particular that if $F = \mathbf{F}_q$ is a finite field then the number of points on the affine part of C_f is divisible by p and the total number of points (including the single point at infinity) equals 1 modulo p .

If $g(x) = f(x) + a^p(x^{pk} - x^k)$ for some $a \in F$ then C_g is F -isomorphic to C_f via the substitution $x, y \rightarrow x, y + ax^k$. If F is a finite field then every element of F is a p -th power and so every curve C_f with $\deg(f) = d$ is isomorphic to a curve C_g s.t. $g = a_d x^d + \dots + a_0$ with $a_{kp} = 0$ for all $k > 0$.

Now assume that $F = \mathbf{F}_q$ is a finite field. An element $a \in F$ can be written as $a = b^p - b, b \in F$ iff $\text{tr}_{F/\mathbf{F}_p} a = 0$ (this follows from the Hilbert 90 theorem or more simply by noting that the map $b \mapsto b^p - b$ is linear with one-dimensional kernel \mathbf{F}_p , while its image is contained in the kernel of the trace map). If $g(x) = f(x) + a$ with $a \in F$ satisfying $\text{tr}_{F/\mathbf{F}_p} a = 0$ then C_g is isomorphic to C_f via $x, y \mapsto x, y + b$, where $a = b^p - b$. Even if $\text{tr}_{F/\mathbf{F}_p} a \neq 0$ the curves C_g becomes isomorphic to C_f over \mathbf{F}_{q^p} (because $\text{tr}_{\mathbf{F}_{q^p}/\mathbf{F}_p} a = 0$). If $g = f + a$ we say that C_g is a twist of C_f .

3.3 Artin-Schreier curves over finite fields and their L-functions

Let $F = \mathbf{F}_q, q = p^n$ be a finite field. Let C be a smooth projective curve over \mathbf{F}_q with genus g . Denote by $N_r(C)$ the number of \mathbf{F}_{q^r} -points on C . The L-function of C is defined by the power series

$$L(z) = \exp \left(\sum_{r=1}^{\infty} \frac{q^r - 1 - N_r(C)}{r} z^r \right). \quad (5)$$

It turns out that $L(z)$ is a polynomial of degree $2g$ and in fact we may write

$$L(z) = \prod_{i=1}^{2g} (1 - \rho_i q^{1/2} z). \quad (6)$$

The normalised zeroes ρ_i come in conjugate pairs and they all satisfy $|\rho_i| = 1$ (the Riemann Hypothesis for curves over a finite field). For all these properties

of the L-function of a curve over a finite field see [15, §5], [17, §V], [13, §3].

Let ζ be a primitive p -th (complex) root of unity. We define an additive character $\psi : \mathbf{F}_p \rightarrow \mathbf{C}^\times$ by $\psi(a) = \zeta^a$ (this is well defined). All the nontrivial additive characters of \mathbf{F}_p are of this form and there are $p - 1$ characters corresponding to the $p - 1$ roots of unity. We denote by $\mathbf{t}_{s/t} : \mathbf{F}_s \rightarrow \mathbf{F}_t$ the trace map from the field with s elements to the field with t elements, provided s is a power of t .

Now let $f \in F[x]$ be a nonconstant polynomial of degree d (we always assume $(d, p) = 1$) and C_f the corresponding A-S curve. Let r be a natural number and $x \in \mathbf{F}_{q^r}$. Any element $a \in \mathbf{F}_{q^r}$ can be written as $a = y^p - y$ with $y \in \mathbf{F}_{q^r}$ iff $\mathbf{t}_{q^r/p} a = 0$ (see previous subsection). Applying this to $f(x)$ we see that C_f has an affine point with abscissa x iff $\mathbf{t}_{q^r/p} f(x) = 0$, in which case it has exactly p such points (namely $(x, y), (x + 1, y), \dots, (x + p - 1, y)$). Using the orthogonality relation for the additive characters of \mathbf{F}_p this can be restated as follows: the number of \mathbf{F}_{q^r} -points on C_f with abscissa $x \in \mathbf{F}_{q^r}$ equals

$$1 + \sum_{\psi \in \Psi} \psi(\mathbf{t}_{q^r/p} x)$$

and so the total number of points on C_f (including the infinite point) is

$$N_r(C_f) = q^r + 1 + \sum_{\alpha \in \mathbf{F}_{q^r}} \sum_{\psi \in \Psi} (\mathbf{t}_{q^r/p} f(\alpha)). \quad (7)$$

Define

$$L_{f,\psi}(z) = \exp \left(\sum_{r=1}^{\infty} \sum_{\alpha \in \mathbf{F}_{q^r}} \psi(\mathbf{t}_{q^r/p} f(\alpha)) \frac{z^r}{r} \right).$$

It follows from (5) and (7) that the L-function of C_f can be written as a product

$$L_f(z) = \prod_{\psi \in \Psi} L_{f,\psi}(z).$$

Each function $L_{f,\psi}$ turns out to be a polynomial of degree $d - 1$ with constant term 1, see [17, §I.3], [13, §4.6.2]. We can write

$$L_{f,\psi} = \prod_{i=1}^{d-1} (1 - \rho_{\psi,i} q^{1/2} z),$$

where the $\rho_{\psi,i}$, $1 \leq i \leq d - 1$, $\psi \in \Psi$ are all the normalised zeroes of $L_f(z)$ and they satisfy $|\rho_{\psi,i}| = 1$. To understand the behaviour of the zeroes of $L_f(z)$ it is enough to study the zeroes of the individual $L_{f,\psi}$ and the relationship between the L-functions corresponding to different characters. These functions are called primitive L-functions or the primitive factors of L_f . Note that for two conjugate characters $\psi, \bar{\psi} \in \Psi$ the two L-functions $L_{f,\psi}, L_{f,\bar{\psi}}$ are conjugate and so are their zeroes. Thus for $p > 2$ the primitive factors come in conjugate

pairs (for $p = 2$ there is only one nontrivial character and L_f itself is a primitive L-function).

Let $f, g \in \mathbf{F}_q[x]$ be polynomials of degree d s.t. $g(x) = f(x) + a(x^{kp} - x^k)$ for some $a \in F$ and natural k . Then for all $\alpha \in \mathbf{F}_{q^r}$ we have $t_{q^r/p}g(\alpha) = t_{q^r/p}f(\alpha)$, since

$$t_{q^r/p}(a(\alpha^{kp} - \alpha^k)) = t_{q/p}(at_{q^r/q}(\alpha^{kp} - \alpha^k)) = t_{q/p}(0) = 0.$$

Therefore $L_{f,\psi} = L_{g,\psi}$. This agrees with the fact that C_f, C_g are isomorphic and so $L_f = L_g$. Now assume that $g(x) = f(x) + a$ for some $a \in \mathbf{F}_q$. From (5) we see that $L_g(z) = L_f(\psi(t_{q/p}a)z)$ and denoting by $\rho_i, \rho'_i, 1 \leq i \leq d-1$ the normalised roots of $L_{f,\psi}, L_{g,\psi}$ respectively we see that $\rho'_i = \psi(-t_{q/p}a)\rho_i$. If $t_{q/p}a = 0$ then $L_{f,\psi} = L_{g,\psi}$ and as we have seen in the previous subsection the curves C_f, C_g are isomorphic in this case.

3.4 The Artin-Schreier family

Denote

$$\mathcal{F}_d = \left\{ f(x) = \sum_{i=0}^d a_i x^i \mid a_i \in \mathbf{F}_q, a_d \neq 0, a_i = 0 \text{ if } p \mid i \text{ and } i \geq 0 \right\}.$$

We refer to $\{C_f\}_{f \in \mathcal{F}_d}$ as the Artin-Schreier (A-S in short) family of curves with parameter d over \mathbf{F}_q and to $\{L_{\psi,f}(z)\}_{f \in \mathcal{F}_d}$ as the A-S family of L-functions with parameter d over \mathbf{F}_q . Note that the latter does not depend on the choice of $\psi \in \Psi$ because for $a \in \mathbf{F}_p^\times$ we have $L_{f,\psi^a}(z) = L_{af,\psi}(z)$ (this follows from (5)), so replacing ψ with ψ^a permutes the family of L-functions.

Fix some element $c \in \mathbf{F}_q$ with $t_{q/p}c = 1$. For $w \in \mathbf{F}_p$ denote $\mathcal{F}_d^w = \{f + wc \mid f \in \mathcal{F}_d\}$. By the observations in the end of the previous subsection we have

$$\{L_{f,\psi}(z) \mid f \in \mathcal{F}_d^w\} = \{L_{f,\psi}(\psi(w)z) \mid f \in \mathcal{F}_d\},$$

so the statistics of zeroes of the L-functions of $f \in \mathcal{F}_d^w$ is essentially the same as that of $f \in \mathcal{F}_d$.

The size of the family \mathcal{F}_d and each \mathcal{F}_d^w is $\#\mathcal{F}_d^w = \#\mathcal{F}_d = q^{d-\lfloor d/p \rfloor}(q-1)$. Denote by \mathcal{F}'_d the set of all degree d polynomials in $\mathbf{F}_q[x]$. Define the map $\mu : \mathcal{F}'_d \rightarrow \cup_{w \in \mathbf{F}_p} \mathcal{F}_d^w$ by

$$\mu \left(\sum_{i=0}^d a_i x^i \right) = (t_{q/p}a_0)c + \sum_{i=1}^d \left(\sum_{j=1}^{\lfloor \log_p(d/i) \rfloor} a_{ipj} \right) x^i.$$

This map is precisely $q^{\lfloor d/p \rfloor + 1}/p$ to one. It follows from the observations in the end of the previous subsection that for any r and any $\alpha \in \mathbf{F}_{q^r}$ we have $t_{q^r/p}(\mu(f)(\alpha)) = t_{q^r/p}f(\alpha)$ and therefore $L(f, \psi) = L(\mu(f), \psi)$.

We conclude that studying the statistics of zeroes of L-functions of $f \in \mathcal{F}'_d$ reduces to studying it for $\mathcal{F}_d^w, w \in \mathbf{F}_p$, which in turn reduces to studying it for the family \mathcal{F}_d . Henceforth we only consider the family \mathcal{F}_d .

4 The random matrix model

In recent decades it has been suggested that the zeroes of L-functions (of all sorts) behave as the spectra of matrices drawn randomly (with some natural measure) from some classical ensemble of matrices. We will illustrate this approach on our example of the A-S L-functions. For the rest of this section assume $p > 2$.

Denote by \mathbf{U}_N the group of $N \times N$ unitary matrices. This is a compact Lie group and so it has a Haar measure. We may draw a random matrix $U \in \mathbf{U}_N$ uniformly w.r.t. the Haar measure and ask about the statistics of its spectrum. The eigenvalues of U lie on the unit circle, just like the zeroes of $L_{f,\psi}$. We take $N = d - 1$ (recall $d = \deg f$) and we model the set of roots of $L_{f,\psi}$ for fixed ψ and "random" f (i.e. f may vary through some large family, e.g. \mathcal{F}_d with either $q \rightarrow \infty$ or $d \rightarrow \infty$, or both) by the spectrum of a random matrix from \mathbf{U}_{d-1} . This model is suggested by the result due to N. Katz and P. Sarnak for the case of fixed d and $q \rightarrow \infty$, stating that the sets of normalised zeroes of the L-functions in the A-S family become equidistributed in the space of sets of eigenvalues of matrices in \mathbf{U}_{d-1} with the measure induced from the Haar measure on U_{d-1} . See Theorem 3.9.2 in [7]. To model the $p - 1$ sets of zeroes of $L_{f,\psi}$, $\psi \in \Psi$ jointly for $p > 2$ we may take $(p - 1)/2$ independent random matrices from \mathbf{U}_{d-1} and their conjugates (see section 3.3).

Now we formulate some conjectures on the statistics of the zeroes of $L_{f,\psi}$, $f \in \mathcal{F}_d$, of which our main theorems are special cases. In all that follows assume $(d, p) = 1$. We are interested in the case where $d \rightarrow \infty$ and q may be fixed or vary as we please. First we consider the linear statistics - the number of zeroes in short intervals and the related quantity of the average of powers of the zeroes. Since multiplying a matrix $U \in \mathbf{U}_N$ by a scalar matrix $e^{i\theta} I_N$ rotates the eigenvalues by an angle of θ it is obvious that the average number of eigenvalues contained in an arc of length l is $Nl/2\pi$ (as U is drawn uniformly at random w.r.t. the Haar measure). We will be interested in $l = O(1/d)$ as $d \rightarrow \infty$, the so-called local regime (it is easier to obtain results for larger arcs). Using our model we formulate

Conjecture 4.1. *Take any $\psi \in \Psi$. Let $C > 0$ be a constant. For every natural number d let I_d be any arc on the unit circle of length C/d . Then the average number of zeroes of $L_{f,\psi}$ contained in I_d as f is chosen uniformly at random from \mathcal{F}_d is $C/2\pi + o(1)$ as $d \rightarrow \infty$.*

Instead of just looking at arcs we may take a smooth window function to count the zeroes. Let $V(t) \in \mathcal{S}(\mathbf{R})$. The function $v_d(t) = \sum_{n=-\infty}^{\infty} V(d(t + 2\pi n))$ is well-defined and periodic with period 2π . It can be viewed as a function on the unit circle. We say that $v_d(t)$ is the periodic window function associated with $V(t)$ with scaling parameter d . For every z with $|z| = 1$ and real number θ the value of $v_d(\arg(z) + \theta)$ is well-defined. Denote $S_f = \sum_{j=1}^{d-1} v_d(\theta_j - \theta)$, where ρ_i are the normalised zeroes of $L_{f,\psi}$. Conjecture 4.1 is equivalent to the following statement: the average of S_f as f is chosen uniformly at random from

\mathcal{F}_d is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} V(t) dt + o(1)$$

as $d \rightarrow \infty$ for any $V \in \mathcal{S}(\mathbf{R})$ and $\theta \in \mathbf{R}$ (because the indicator of an interval can be approximated by a window function in $\mathcal{S}(\mathbf{R})$ and any window function can be approximated by a superposition of interval indicators).

Now we consider the quantity $T_{f,\psi}^r = \sum_{i=1}^{d-1} \rho_{f,\psi,i}^r$, where as usual $\rho_{f,\psi,i}$ are the normalised zeroes of $L_{f,\psi}$. The uniform distribution of the L-zeroes on the unit circle suggests the following

Conjecture 4.2. *Take any $\psi \in \Psi$. For every $\epsilon > 0$ the average of $T_{f,\psi}^r$ where f is drawn uniformly at random from \mathcal{F}_d is*

$$O_\epsilon \left(q^{\epsilon r + (1/p-1/2)d} \right)$$

as $d \rightarrow \infty$ and $r \geq d$ (q may vary with d as we please).

It can be shown by a standard argument that Conjecture 4.2 combined with Theorem 1 implies Conjecture 4.1. See the proof of Theorem 2 in section 5.2 for this kind of argument. We remark that Conjecture 4.2 would follow from a function field analogue of a conjecture of H. Montgomery about the distribution of primes in arithmetic progressions (see [12, §13.1]).

At this point a simpler model for the zeroes of L-functions in the A-S family would be just $d-1$ independent random points on the unit circle (with uniform distribution), which is also consistent with Conjectures 4.1, 4.2 and Theorems 1, 2. However in section 6 we study nonlinear statistics of the zeroes which show agreement with the random unitary matrix model and disagreement with the independent random points model.

Finally we note that for $p = 2$ we need a different model, namely a random symplectic matrix model. See section 5.3 for a description of this model and some partial results.

5 Proof of the main results

5.1 Proof of Theorem 1

We keep the notation of the previous section. Let p, q be as in section 1, $\psi \in \Psi$, d a natural number satisfying $(d, p) = 1$. First we need a lemma

Lemma 5.1. *Let $\psi \in \Psi$ be a character, $f \in \mathbf{F}_q[x]$ of degree d . Let $\rho_i, 1 \leq i \leq d-1$ be the normalised zeroes of $L_{f,\psi}$ and $T_{f,\psi}^r = \sum_{i=1}^{d-1} \rho_i^r$. Then*

$$T_{f,\psi}^r = -q^{-r/2} \sum_{\alpha \in \mathbf{F}_{q^r}} \psi(\mathfrak{t}_{q^r/p} f(\alpha)).$$

Proof. This is a well known fact that follows directly from (5) and (6). See [17, §I.3.3] for details. \square

For the rest of this section fix $\psi \in \Psi$. To prove Theorem 1 we need to estimate the average of the sum $\sum_{\alpha \in \mathbf{F}_{q^r}} \psi(t_{q^r/p} f(\alpha))$ as f varies uniformly through \mathcal{F}_d . Recall that \mathcal{F}_d consists of the degree d polynomials $f = \sum_{i=0}^d a_i x^i$ with $a_d \neq 0$ and $a_{kp} = 0$ for all $k \geq 0$.

We begin with a simple observation that establishes a weak form of Theorem 1, namely with $r < d$.

Lemma 5.2. *Assume $r < d$. Let $\alpha \in \mathbf{F}_{q^r}$ be an element. Then*

$$\langle \psi(t_{q^r/p} f(\alpha)) \rangle_{f \in \mathcal{F}_d} = \begin{cases} 1, & \alpha = 0 \text{ or } p|r, \alpha \in \mathbf{F}_{q^{r/p}}, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Proof. If $\alpha = 0$ the assertion is clear since $f(\alpha) = f(0) = 0$ and so $\psi(t_{q^r/p} f(\alpha)) = 1$ for all $f \in \mathcal{F}_d$. If $p|r$ and $\alpha \in \mathbf{F}_{q^{r/p}}$ then for all $f \in \mathcal{F}_d$ we have $f(\alpha) \in \mathbf{F}_{q^{r/p}}$, so $t_{q^r/p} f(\alpha) = p \cdot t_{q^r/p/p} f(\alpha) = 0$ and $\psi(t_{q^r/p} f(\alpha)) = 1$.

Now assume that $\alpha \neq 0$ and if $p|r$ then $\alpha \notin \mathbf{F}_{q^{r/p}}$. This means that the minimal polynomial h of α over \mathbf{F}_q satisfies $(r/\deg h, p) = 1$. Denote $\mathcal{F}_d'' = \{f \in \mathbf{F}_q[x] \mid \deg f = d, f(0) = 0\}$. Recall the definition of the map μ in section 3.4. Since the $\mu|_{\mathcal{F}_d''} : \mathcal{F}_d'' \rightarrow \mathcal{F}_d$ is precisely $q^{\lfloor d/p \rfloor}$ to one and preserves $t_{q^r/p} f(\alpha)$ so we may replace \mathcal{F}_d by \mathcal{F}_d'' in (8). Let $h \in \mathbf{F}_q[x]$ be the minimal polynomial of α over \mathbf{F}_q . Since $d > r$ the map $\pi : \mathcal{F}_d'' \rightarrow \mathbf{F}_q[x]/h \cong \mathbf{F}_{q^r}$ defined by $\pi(f) = (f/x) \bmod h$ is exactly $(q-1)q^{d-r-1}$ to one and since x is invertible modulo h (as $\alpha \neq 0$) so is the map $\pi' : \mathcal{F}_d'' \rightarrow \mathbf{F}_{q^r} \cong \mathbf{F}_q[x]/h$ defined by $\pi'(f) = f(\alpha)$. Thus each value of $f(\alpha) \in \mathbf{F}_{q^{\deg h}} \cong \mathbf{F}_q[x]/h$ is obtained equally many times as f ranges through \mathcal{F}_d'' . Since $(r/\deg h, p) = 1$ the value of $\psi(t_{q^r/p} \gamma) = \psi(t_{q^{\deg h}/p} \gamma)^{r/\deg h}$ is uniformly distributed among the p -th roots of unity as γ ranges through $\mathbf{F}_{q^{\deg h}}$, which proves the assertion of the lemma. \square

The following corollary establishes Theorem 1 for $r < d$.

Corollary 5.3. *Assume $r < d$. Then $M_d^r = -e_{p,r} q^{r/p-r/2} + (e_{p,r} - 1) q^{-r/2}$.*

Proof. By Lemma 5.1 we have

$$M_d^r = \langle T_{f,\psi}^r \rangle_{f \in \mathcal{F}_d} = -q^{-r/2} \sum_{\alpha \in \mathbf{F}_{q^r}} \langle \psi(t_{q^r/p} f(\alpha)) \rangle_{f \in \mathcal{F}_d}. \quad (9)$$

Now using Lemma 5.2 we see that the RHS of (9) equals $q^{-r/2}$ if $(r, p) = 1$ (only $\alpha = 0$ contributes 1 to the sum) or $q^{r/p-r/2}$ if $p|r$ (each $\alpha \in \mathbf{F}_{q^r}$ contributes 1 to the sum). \square

To go further we need some lemmata.

Lemma 5.4. *Let $\alpha \in \mathbf{F}_{q^r}$ have monic minimal polynomial $h(x) = \sum_{i=0}^r c_r x^r \in \mathbf{F}_q[x]$ of degree r . Assume that for some $0 < k \leq r$ with $(k, p) = 1$ we have $c_{r-k} \neq 0$. Then for the minimal such k we have*

$$t_{q^r/q}(\alpha^k) = -k c_{r-k} \neq 0$$

and for all $0 < j < k$ we have

$$\sum_{i=1}^r \alpha_i^j = 0.$$

Proof. Denote by $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_r$ the conjugates of α over \mathbf{F}_q . Denote by $\sigma_i(x_1, \dots, x_r)$ the degree i elementary symmetric polynomial in r variables. We have $c_{r-i} = (-1)^i \sigma_i(\alpha_1, \dots, \alpha_r)$ for all $1 \leq i \leq r$. Denote $s_i(x_1, \dots, x_r) = \sum_{j=1}^r x_j^i$. Newton's identity (see [14, §3.1.1]) states that for all $1 \leq m \leq r$ we have

$$m\sigma_m = (-1)^{m+1} s_m - \sum_{i=1}^{m-1} s_i \sigma_{m-i}. \quad (10)$$

We show by induction that for $1 \leq i < k$ we have $s_i(\alpha_1, \dots, \alpha_r) = 0$. If $k > 1$ then the case $i = 1$ is clear as by assumption $\sigma_1(\alpha_1, \dots, \alpha_i) = -c_{r-1} = 0$. Assume that $i < k$ and that $s_j(\alpha_1, \dots, \alpha_r) = 0$ holds for all $1 \leq j < i$. By (10) we have

$$ic_{r-i} = \pm i \sigma(\alpha_1, \dots, \alpha_r) = \pm s_i(\alpha_1, \dots, \alpha_r)$$

(the other terms in the identity are zero by the induction hypothesis). Now if i is not divisible by p the assumption on k implies that $c_{r-i} = 0$ and if i is divisible by p we still have $ic_{r-i} = 0$ (as we are in characteristic p). This completes the induction. Now again we see from (10) that

$$s_k(\alpha_1, \dots, \alpha_r) = (-1)^{k+1} k \sigma_k(\alpha_1, \dots, \alpha_r) = -k c_{r-k}$$

as required. \square

Lemma 5.5. *Let $\alpha \in \mathbf{F}_{q^r}$ have monic minimal polynomial $h(x) = \sum_{i=0}^r c_i x^i \in \mathbf{F}_q[x]$ of degree r . Assume that either $r < d$ or $r \geq d$ and for some $0 \leq k < d$ with $(k, p) = 1$ we have $c_{r-k} \neq 0$. Then $\langle \psi(\text{tr}_{q^r/p} f(\alpha)) \rangle_{f \in \mathcal{F}_d} = 1$. If $d \geq r$ and there is no such k then if $c_{r-d} = 0$ we have $\langle \psi(\text{tr}_{q^r/p} f(\alpha)) \rangle_{f \in \mathcal{F}_d} = 1$ and if $c_{r-d} \neq 0$ we have $\langle \psi(\text{tr}_{q^r/p} f(\alpha)) \rangle_{f \in \mathcal{F}_d} = -1/(q-1)$.*

Proof. The case $r \leq d$ follows from Lemma 5.2, so we assume $d < r$. First assume there exists $0 \leq k < d$ s.t. $(k, p) = 1$ and $c_{r-k} \neq 0$. Let k be minimal with this property. By the previous lemma $\text{t}_{q^r/q}(\alpha^k) = -k c_{r-k} \neq 0$. Therefore there exists $a \in \mathbf{F}_q$ s.t.

$$\text{t}_{q^r/p}(a\alpha^k) = \text{t}_{q^r/q}(a\text{t}_{q/p}(\alpha^k)) \neq 0.$$

Now the set \mathcal{F}_d can be partitioned into subsets of the form

$$S_g = \{g, g + ax^k, g + 2ax^k, \dots, g + (p-1)ax^k\}.$$

Note however that

$$\begin{aligned} \sum_{f \in S_g} \psi(\text{tr}_{q^r/p} f(\alpha)) &= \psi(\text{tr}_{q^r/p} g(\alpha)) \sum_{i=0}^{p-1} \psi(\text{tr}_{q^r/p}(ia\alpha^k)) = \\ &= \psi(\text{tr}_{q^r/p} g(\alpha)) \sum_{i=1}^{p-1} \psi(\text{tr}_{q^r/p}(a\alpha^k))^i = 0 \end{aligned}$$

because $\psi(\text{tr}_{q^r/p}(a\alpha^k))$ is a primitive p -th root of unity. Since \mathcal{F}_d is partitioned into sets of the form S_g we get the first claim of the lemma.

Now assume that for all $0 \leq k < d, (k, p) = 1$ we have $c_{r-k} = 0$. Take some $\alpha \in \mathbf{F}_{q^r}$ of degree r . By the previous lemma we get that $\text{tr}_{q^r/q}(\alpha^i) = 0$ for $0 \leq i < d$ and so $\text{t}_{q^r/p}(a\alpha^i) = 0$ for all $a \in \mathbf{F}_q$ and by the second part of the lemma we have $\text{t}_{q^r/q}(\alpha^d) = -dc_{r-d}$ and so $\text{t}_{q^r/p}(a\alpha^d) = -d\text{t}_{q/p}(ac_{r-d})$ for every $a \in \mathbf{F}_q$. Thus for every $f = \sum_{i=0}^d \in \mathcal{F}_d$ we have $\text{t}_{q^r/p}f(\alpha) = -d\text{t}_{q/p}(a_dc_{r-d})$.

If $c_{r-d} = 0$ then $\langle \psi(\text{t}_{q^r/p}f(\alpha)) \rangle_{f \in \mathcal{F}_d} = 1$. Assume $c_{r-d} \neq 0$. The leading coefficient of $f \in \mathcal{F}_d$ is distributed uniformly in $\mathbf{F}_q^\times = \mathbf{F}_q \setminus \{0\}$ and so is a_dc_{r-d} . We have

$$\langle \psi(\text{t}_{q/p}a) \rangle_{a \in \mathbf{F}_q^\times} = -1/(q-1)$$

because as a ranges through \mathbf{F}_q^\times each nonzero value of $\text{t}_{q/p}a$ occurs q/p and zero occurs $q/p - 1$ times, so $\sum_{a \in \mathbf{F}_q^\times} \psi(a) = -1$. This concludes the proof of the lemma. \square

For $r > d$ denote by $\eta_d(r)$ the number of monic irreducible polynomials $h(x) = x^r + \sum_{i=0}^{r-1} c_i x^i$ s.t. $c_{r-k} = 0$ for all $1 \leq k < d$ with $(k, p) = 1$. Denote by $\eta_d^0(r)$ the number of such polynomials with $c_{r-d} = 0$ (if $r = d$ we define $\eta_d^0(r) = 0$).

Proposition 5.6.

$$M_d^r = \frac{q^{-r/2+1}}{q-1} \sum_{s|r, (r/s, p)=1, s \geq d} s(\eta_d(s)/q - \eta_d^0(s)) - e_{p,r} q^{r/p-r/2} + (e_{p,r} - 1)q^{-r/2}.$$

Proof. By Lemma 5.1 we have

$$M_d^r = q^{-r/2} \sum_{\alpha \in \mathbf{F}_{q^r}} \langle \psi(\text{t}_{q^r/p}f(\alpha)) \rangle_{f \in \mathcal{F}_d}. \quad (11)$$

Every $\alpha \in \mathbf{F}_{q^r}$ has degree $s|r$ over \mathbf{F}_q . First let $s|r$ be such that $(r/s, p) = 1$. For α of degree s (over \mathbf{F}_q) and $f \in \mathcal{F}_d$ we have $\text{t}_{q^r/p}f(\alpha) = (r/s)\text{t}_{q^s/p}f(\alpha)$ and so

$$\psi(\text{t}_{q^r/p}f(\alpha)) = \psi^{r/s}(\text{t}_{q^s/p}f(\alpha)).$$

By Lemma 5.5 applied to $s, \psi^{r/s}$ instead of r, ψ we see that the contribution of all $\alpha \neq 0$ of degree s to the RHS of (11) is

$$\frac{q^{-r/2+1}}{q-1} s(\eta_d(s)/q - \eta_d^0(s)),$$

since each irreducible polynomial $h = \sum c_i x^i$ of degree $s > d$ has s roots, each contributing 1 to the sum if $c_{s-d} = 0$ and $-1/(q-1)$ otherwise (elements of degree $s \leq d$ contribute nothing by Lemma 5.2). If $(r, p) = 1$ we obtain the assertion of the lemma. It remains to consider the contribution of $\alpha = 0$ and $\alpha \in \mathbf{F}_{q^{r/p}}$ in case that $p|r$. Since for $\alpha = 0$ and $\alpha \in \mathbf{F}_{q^{r/p}}$ we have $\text{t}_{q^r/p}f(\alpha) = 0$, this contribution is obviously $-e_{p,r} q^{r/p-r/2} + (e_{p,r} - 1)q^{-r/2}$. \square

Theorem 1 follows at once from proposition 5.6. Indeed the total number of polynomials of the form $h(x) = x^s + \sum_{i=0}^{s-1} c_i x^i$ with $s \geq d$, $s|r$ and $c_{s-kp} = 0$, $1 \leq k \leq \lfloor d/p \rfloor$ is at most $O(q^{r-d+\lfloor d/p \rfloor})$ and therefore

$$M_{d,\psi}^r = -e_{p,r} q^{r/p-r/2} + O\left(r q^{r/2-(1-1/p)d} + q^{-r/2}\right).$$

Note that the assertion of the theorem is interesting only for $r < 2d(1 - 1/p)$, because by the definition of $T_{d,\psi}^f$ (see Lemma 5.1) we have $T_{d,\psi}^f = O(d)$ for $f \in \mathcal{F}_d$ and so $M_{d,\psi}^r = O(d)$.

5.2 Proof of Theorem 2

Now we deduce Theorem 2 from Theorem 1. For this subsection we assume $p > 2$. Let $V \in \mathcal{S}(\mathbf{R})$ be a fixed window function. From a window function W we may construct a periodic window function with parameter d (natural number) as follows:

$$v_d(t) = \sum_{r=-\infty}^{\infty} V(d(t + 2\pi r)).$$

We also define $v_{d,\theta}(t) = v_d(t - \theta)$. The function $v_{d,\theta}$ has period 2π and as $d \rightarrow \infty$ it becomes "localised" at points of the form $\theta + 2\pi m$, $m \in \mathbf{Z}$. The Fourier transform of $V(t)$ is given by

$$\hat{V}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(t) e^{-ist} dt$$

and the r -th Fourier coefficient of $v_{d,\theta}$ is

$$\hat{v}_{d,\theta}(r) = \int_0^{2\pi} v_{d,\theta}(t) e^{-irt} dt.$$

A simple calculation shows that

$$\hat{v}_{d,\theta}(r) = \frac{e^{-ir\theta}}{d} \hat{V}\left(\frac{r}{d}\right). \quad (12)$$

Lemma 5.7. *For $f \in \mathcal{F}_d$ denote*

$$S_f = \sum_{j=1}^{d-1} v_{d,\theta}(\theta_j),$$

where $\rho_j = e^{i\theta_j}$ are the normalised zeroes of $L_{f,\psi}$ (θ_j are real numbers well defined modulo 2π). Then

$$S_f = \hat{V}(0) + \sum_{r=1}^{\infty} \left(\hat{V}\left(\frac{r}{d}\right) \frac{e^{-ir\theta}}{d} T_{f,\psi} + \hat{V}\left(-\frac{r}{d}\right) \frac{e^{ir\theta}}{d} \overline{T_{f,\psi}^r} \right).$$

Proof. Since $V \in \mathcal{S}(\mathbf{R})$ the function $v_{d,\theta}$ is smooth and so for $|z| = 1$ we have

$$v_{d,\theta}(\arg z) = \sum_{r=-\infty}^{\infty} \hat{v}_{d,\theta}(r) z^r.$$

Applying this to $z = \rho_j$, noting that $\rho_j^{-r} = \bar{\rho}_j^r$, using (12) and summing over i we obtain the assertion of the lemma. \square

Corollary 5.8.

$$\langle S_f \rangle_{f \in \mathcal{F}_d} = \hat{V}(0) + \sum_{r=1}^{\infty} \left(\hat{V}\left(\frac{r}{d}\right) e^{-ir\theta} + \hat{V}\left(-\frac{r}{d}\right) e^{ir\theta} \right) \frac{M_d^r}{d}.$$

Proof. Just average the previous lemma over $f \in \mathcal{F}_d$ and note that $M_d^r \in \mathbf{R}$ by Proposition 5.6. \square

Now we are ready to prove Theorem 2. Assume that \hat{V} is supported in $(-(2-2/p), 2-2/p)$. There exists $\epsilon > 0$ s.t. $\hat{V}(r/d) = 0$ for all $r \geq (2-2/p-\epsilon)d$. Using the last corollary and Theorem 1 we obtain

$$\begin{aligned} \langle S_f \rangle_{f \in \mathcal{F}_d} &= \hat{V}(0) + \sum_{r=1}^{\lfloor (2-2/p-\epsilon)d \rfloor} \left(\hat{V}\left(\frac{r}{d}\right) e^{-ir\theta} + \hat{V}\left(-\frac{r}{d}\right) e^{ir\theta} \right) \frac{M_d^r}{d} = \\ &= \hat{V}(0) + \sum_{r=1}^{\lfloor (2-2/p-\epsilon)d \rfloor} O\left(q^{r/p-r/2} + rq^{r/2-(1-1/p)d}\right) \frac{1}{d} = \\ &= \hat{V}(0) + O(1/d) + O\left(\frac{q^{-\epsilon d}}{d}\right) = \hat{V}(0) + o(1) \end{aligned}$$

as $d \rightarrow \infty$ (note that we used the fact that $\hat{V} = O(1)$ since $V \in \mathcal{S}(\mathbf{R})$ is fixed). It is now enough to notice that $\hat{V}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(t) dt$.

Remark. As can be seen from the above proof, the $o(1)$ term in Theorem 2 can be replaced with $O(1/d)$.

5.3 $p=2$

If $p = 2$ the assertion of Theorem 2 does not hold and has to be modified. The random matrix model needs to be modified as well. Indeed there is only one character $\psi \in \Psi$ and the L-function $L_f = L_{f,\psi}$ for $f \in \mathcal{F}_d$ is already primitive. Since it has real coefficients it is obvious that the set of normalised zeroes consists of conjugate pairs and cannot be "random" in the space of eigenvalue sets of unitary matrices. Instead we should consider a random unitary symplectic matrix $U \in \mathbf{USp}(d-1)$ (thus we denote the group of $(d-1) \times (d-1)$ unitary symplectic matrices). In fact for $p = 2$ the curves in the A-S family are hyperelliptic and the usual model for families of hyperelliptic curves is the random symplectic matrix model, see for example the work mentioned in section 2.1.

For the unitary symplectic group the following holds:

$$\langle T_U^r \rangle_{U \in \mathbf{USp}(d-1)} = \begin{cases} -e_{2,r}, & r < d, \\ 0, & r \geq d, \end{cases}$$

see [4, §4]. From this and Theorem 1 we conclude that for $r < d$ we have

$$M_d^r = \langle T_U^r \rangle_{U \in \mathbf{USp}(d-1)} + O\left(q^{-r/2}\right).$$

From this one can derive using the method of section 5.2 the following

Corollary 5.9. *Let $V \in \mathcal{S}(\mathbf{R})$ be a window function,*

$$v_d(t) = \sum_{n=-\infty}^{\infty} V(d(t + 2\pi n)).$$

For $f \in \mathcal{O}_d$ denote $Z_f = \sum_{j=1}^{d-1} v_d(\theta_j)$, where $\rho_j = e^{i\theta_j}$ are the normalised zeroes of L_f . Similarly for a matrix $U \in \mathbf{USp}(d-1)$ with eigenvalues $\rho_j = e^{i\theta_j}$ denote $Z_U = \sum_{j=1}^{d-1} v_d(\theta_j)$. Assume that \hat{V} is supported on $[-1, 1]$. Then

$$\langle Z_f^r \rangle_{f \in \mathcal{F}_d} \rightarrow \langle Z_U^r \rangle_{U \in \mathbf{USp}(d-1)}$$

as $d \rightarrow \infty$.

6 Nonlinear statistics

Theorems 1 and 2 suggest that the zeroes of a random L-function from the A-S family are (at least on average) rather uniformly distributed on the unit circle, but this seems like a weak confirmation of the random unitary matrix model. A simpler model would be $d-1$ independent random points on the circle (with uniform distribution). In this section we study more delicate statistics of the zeroes and show agreement with the random unitary matrix model and disagreement with the independent random points model.

We preserve the notation of the previous sections. Let $V(t, u) \in \mathcal{S}(\mathbf{R}^2)$ be a two-variable window function, N a natural number and $v_N(t, u)$ the periodic window function associated with V by

$$v_N(t, u) = \sum_{m, n=-\infty}^{\infty} V(N(t + 2\pi m), N(u + 2\pi n)).$$

Let ρ_1, \dots, ρ_N be n points on the unit circle, $\rho_j = e^{i\theta_j}$. Finally let θ be some fixed real number. We consider the 2-level density function (at θ):

$$S_\theta^2(\rho_1, \dots, \rho_N) = \sum_{\substack{1 \leq j, k \leq N \\ j \neq k}} v_N(\theta_j - \theta, \theta_k - \theta).$$

For a matrix $U \in \mathbf{U}(N)$ with eigenvalues ρ_1, \dots, ρ_N we denote

$$S_\theta^2(U) = S_\theta^2(\rho_1, \dots, \rho_N)$$

and for an A-S L-function $L_{f,\psi}$ with normalised zeroes $\rho_1, \dots, \rho_{d-1}$ we denote

$$S_\theta^2(f, \psi) = S_\theta^2(\rho_1, \dots, \rho_{d-1}).$$

It is easy to see that if ρ_1, \dots, ρ_N are selected uniformly and independently on the unit circle, then the average of $S_\theta^2(\rho_1, \dots, \rho_N)$ tends to

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(t, u) dt du$$

as $N \rightarrow \infty$ (for any θ). On the other hand we have the following result (see [8, §AD.2]):

$$\langle S_\theta^2(U) \rangle_{U \in \mathbf{U}(N)} \rightarrow \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(t, u) \left(1 - \left(\frac{\sin(\pi(t-u))}{\pi(t-u)} \right)^2 \right) dt du. \quad (13)$$

as $N \rightarrow \infty$ (the average is taken w.r.t. the Haar measure), for any θ . Theorem 3, which we prove in the present section, provides evidence for the random unitary matrix model.

6.1 Product of traces

For the rest of section 6 we assume that $p > 2$. Just as we used the quantities $T_{f,\psi}^r, M_d^r$ to study the linear statistics of the zeroes of $L_{f,\psi}$ we introduce the quantities

$$T_{f,\psi}^{r,s} = T_{f,\psi}^r T_{f,\psi}^s, M_d^{r,s} = \langle T_{f,\psi}^{r,s} \rangle_{f \in \mathcal{F}_d}$$

(again $M_d^{r,s}$ does not depend on the choice of $\psi \in \Psi$). We also define $T_{f,\psi}^r$ for any integer r (possibly negative) by the same expression $T_{f,\psi}^r = \sum_{j=1}^{d-1} \rho_j^r$ (where ρ_j are the normalised zeroes of $L_{f,\psi}$) and extend the definition of $M_d^r, T_{f,\psi}^{r,s}, M_d^{r,s}$ to all integers r, s . Note that $T_{f,\psi}^{-r} = \overline{T_{f,\psi}^r}$. Good estimates for these quantities provide good estimates for the quadratic statistics of the L-zeroes, such as the square of the number of points in short intervals and the 2-level density.

Lemma 6.1. *For $r, s > 0$ we have*

$$T_{f,\psi}^{r,s} = q^{-(r+s)/2} \sum_{\alpha \in \mathbf{F}_{q^r}, \beta \in \mathbf{F}_{q^s}} \psi(t_{q^r/p} f(\alpha) + t_{q^s/p} f(\beta)),$$

$$T_{f,\psi}^{r,-s} = q^{-(r+s)/2} \sum_{\alpha \in \mathbf{F}_{q^r}, \beta \in \mathbf{F}_{q^s}} \psi(t_{q^r/p} f(\alpha) - t_{q^s/p} f(\beta)).$$

Proof. This follows immediately from Lemma 5.2. \square

Lemma 6.2. Assume $r, s > 0$, $r + s < d$. Let $\alpha \in \mathbf{F}_{q^r}, \beta \in \mathbf{F}_{q^s}$ be nonzero elements with monic minimal polynomials g, h over \mathbf{F}_q respectively. For any natural m denote

$$A_m = \begin{cases} \mathbf{F}_{q^{m/p}}, & p|m, \\ \{0\}, & p \nmid m. \end{cases}$$

We have

$$\langle \psi(\mathfrak{t}_{q^r/p}f(\alpha) - \mathfrak{t}_{q^s/p}f(\beta)) \rangle_{f \in \mathcal{F}_d} = \begin{cases} 1, & g = h, p \deg g | r - s \text{ or } \alpha \in A_r, \beta \in A_s, \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

$$\langle \psi(\mathfrak{t}_{q^r/p}f(\alpha) + \mathfrak{t}_{q^s/p}f(\beta)) \rangle_{f \in \mathcal{F}_d} = \begin{cases} 1, & g = h, p \deg g | r + s \text{ or } \alpha \in A_r, \beta \in A_s, \\ 0, & \text{otherwise.} \end{cases} \quad (15)$$

Proof. We prove (14), (15) being similar. If $g = h$ then α, β are conjugate over \mathbf{F}_q and so are $f(\alpha), f(\beta)$, so $\mathfrak{t}_{q^m/p}f(\alpha) = \mathfrak{t}_{q^m/p}f(\beta)$, where $m = \deg g$. We have

$$\mathfrak{t}_{q^r/p}f(\alpha) = \frac{r}{m} \mathfrak{t}_{q^m/p}f(\alpha), \mathfrak{t}_{q^s/p}f(\beta) = \frac{s}{m} \mathfrak{t}_{q^m/p}f(\beta)$$

and so

$$\mathfrak{t}_{q^r/p}f(\alpha) - \mathfrak{t}_{q^s/p}f(\beta) = \frac{r-s}{m} \mathfrak{t}_{q^m/p}f(\alpha).$$

If $pm|r-s$ then $\frac{r-s}{m}$ is divisible by p and so $\mathfrak{t}_{q^r/p}f(\alpha) - \mathfrak{t}_{q^s/p}f(\beta) = 0$ and $\psi(\mathfrak{t}_{q^r/p}f(\alpha) - \mathfrak{t}_{q^s/p}f(\beta)) = 1$ for all $f \in \mathcal{F}_d$ and of course it also holds on average. If $pm \nmid r-s$ then denoting $l = (r-s)/m \bmod p$ we get from Lemma 5.2 applied to the nontrivial character ψ^l instead of ψ that the LHS of 14 equals 0.

For $\alpha \in A_r$ we have $\mathfrak{t}_{q^r/p}f(\alpha) = 0$ because if $p|r$ we have $\mathfrak{t}_{q^r/p}f(\alpha) = p \cdot \mathfrak{t}_{q^{r/p}/p}f(\alpha) = 0$ and if $p \nmid r$ then $\alpha = 0$ and $f(\alpha) = 0$ for all $f \in \mathcal{F}_d$. Similarly if $\beta \in A_s$ then $\mathfrak{t}_{q^s/p}f(\beta) = 0$. We see that if $\alpha \in A_r, \beta \in A_s$ then $\psi(\mathfrak{t}_{q^r/p}f(\alpha) - \mathfrak{t}_{q^s/p}f(\beta)) = 1$ for all $f \in \mathcal{F}_d$ and the same holds for the average.

Now assume that $\beta \in A_s$ but $\alpha \notin A_r$. Then for all $f \in \mathcal{F}_d$ we have $\psi(\mathfrak{t}_{q^r/p}f(\alpha) - \mathfrak{t}_{q^s/p}f(\beta)) = \psi(\mathfrak{t}_{q^r/p}f(\alpha))$ and so

$$\langle \psi(\mathfrak{t}_{q^r/p}f(\alpha) - \mathfrak{t}_{q^s/p}f(\beta)) \rangle_{f \in \mathcal{F}_d} = \langle \psi(\mathfrak{t}_{q^r/p}f(\alpha)) \rangle_{f \in \mathcal{F}_d} = 0$$

by Lemma 5.2, since $s < d$ and $\alpha \notin A_r$. The case $\alpha \in A_r, \beta \notin A_s$ is treated similarly.

Finally assume that $g \neq h$ and $\alpha \notin A_r, \beta \notin A_s$, i.e. $\alpha, \beta \neq 0, pu \nmid r, pv \nmid s$, where $u = \deg g, v = \deg h$ (we have $u|r, v|s$). As in the proof of Lemma 5.2 we may average over

$$\mathcal{F}_d'' = \{f \in \mathbf{F}_q[x] \mid \deg f = d, f(0) = 0\}$$

instead of \mathcal{F}_d (this does not change the average). Now since $u + v \leq r + s < d$, the map $\pi : \mathcal{F}_d'' \rightarrow \mathbf{F}_q[x]/gh$ defined by $\pi(f) = (f/x) \bmod h$ is exactly $(q-1)q^{d-u-v-1}$ to one and since x is invertible modulo gh (as $\alpha, \beta \neq 0$) so is

the map $\pi' : \mathcal{F}_d'' \rightarrow \mathbf{F}_q[x]/gh$ defined by $\pi'(f) = f \bmod gh$. However by the Chinese remainder theorem we have $\mathbf{F}_q[x]/gh \cong \mathbf{F}_q[x]/g \times \mathbf{F}_q[x]/h \cong \mathbf{F}_{q^u} \times \mathbf{F}_{q^v}$ (direct product of rings) with the isomorphism given by $f \mapsto (f(\alpha), f(\beta))$. We conclude that as f ranges over \mathcal{F}_d'' each pair $(f(\alpha), f(\beta)) \in \mathbf{F}_{q^u} \times \mathbf{F}_{q^v}$ is obtained equally many times. However since $pu \nmid r$ the map $\sigma : \mathbf{F}_{q^u} \rightarrow \mathbf{F}_p$ defined by $\sigma(\gamma) = \psi(t_{q^r/p}\gamma) = \psi(t_{q^u/p}\gamma)^{r/u}$ also assumes every value equally many times and the same goes for $\psi(t_{q^r/p}\gamma)$ on \mathbf{F}_{q^v} . We conclude that as f ranges over \mathcal{F}_d'' each p -th root of unity occurs equally many times as $\psi(t_{q^r/p}f(\alpha) - t_{q^r/p}f(\beta))$ and since averaging over \mathcal{F}_d is equivalent to averaging over \mathcal{F}_d'' we obtain (14). \square

Now denote by $\pi(m)$ the number of monic irreducible polynomials in $\mathbf{F}_q[x]$ with degree m .

Proposition 6.3. *Assume $r, s > 0$ and $r + s < d$. Then*

$$M_d^{r,-s} = q^{-(r+s)/2} \left(\sum_{\substack{m|(r,s) \\ mp|r-s \\ mp \nmid r}} \pi(m)m^2 + e_{p,r}e_{p,s}q^{(r+s)/p} \right) + \\ + q^{-(r+s)/2} \left((1 - e_{p,r})e_{p,s}q^{s/p} + (1 - e_{p,s})e_{p,r}q^{r/p} + (1 - e_{p,r})(1 - e_{p,s}) \right), \quad (16)$$

$$M_d^{r,s} = q^{-(r+s)/2} \left(\sum_{\substack{m|(r,s) \\ mp|r+s \\ mp \nmid r}} \pi(m)m^2 + e_{p,r}e_{p,s}q^{(r+s)/p} \right) + \\ + q^{-(r+s)/2} \left((1 - e_{p,r})e_{p,s}q^{s/p} + (1 - e_{p,s})e_{p,r}q^{r/p} + (1 - e_{p,r})(1 - e_{p,s}) \right). \quad (17)$$

Proof. We prove (16), (17) begin similar. By Lemma 6.1 we have

$$M_d^{r,-s} = q^{-(r+s)/2} \sum_{\substack{\alpha \in \mathbf{F}_{q^r} \\ \beta \in \mathbf{F}_{q^s}}} \langle \psi(t_{q^r/p}f(\alpha) - t_{q^s/p}f(\beta)) \rangle_{f \in \mathcal{F}_d}. \quad (18)$$

Now we can use Lemma 6.2 to evaluate this expression. Denote by g, h the monic minimal polynomials (over \mathbf{F}_q) of $\alpha \in \mathbf{F}_{q^r}, \beta \in \mathbf{F}_{q^s}$ respectively. First we count the contribution of those α, β for which $g = h$ (i.e. they are conjugate), $p \deg g | r - s$ and $p \deg g \nmid r$ (and consequently $p \deg g \nmid s$), so $\alpha \notin A_r$ and $\beta \notin A_s$. Each polynomial g has exactly $\deg g$ roots and the number of pairs α, β with minimal polynomial g is $(\deg g)^2$. The contribution of all such g to the sum in (18) is

$$\sum_{\substack{m|(r,s) \\ mp|r-s \\ mp \nmid r}} \pi(m)m^2.$$

It remains to evaluate the contribution of the pairs $\alpha \in A_r, \beta \in A_s$ (see the notation in the previous lemma). If $p|(r, s)$ then every pair $\alpha \in \mathbf{F}_{q^{r/p}}, \beta \in \mathbf{F}_{q^{s/p}}$ contributes 1 to the sum (by Lemma 6.2) and the number of such pairs is $q^{(r+s)/p}$. If $p|r$ but $p \nmid s$ then the pairs $\alpha \in \mathbf{F}_{q^{r/p}}, \beta = 0$ (and only them) contribute 1, so we get a total contribution of $q^{r/p}$. The case $p \nmid r, p|s$ is treated similarly. If $p \nmid rs$ then only $\alpha = \beta = 0$ adds 1 to the sum. In any case we get the value stated in the proposition.

To prove (17) we proceed similarly using (15). \square

Theorem 9. *Assume $r \geq s > 0$ and $r + s < d$. Then*

$$M_d^{r,-s} = \delta_{r,s}r + O\left(rq^{-r/2} + q^{(1/p-1/2)(r+s)}\right),$$

where

$$\delta_{r,s} = \begin{cases} 1, & r = s, \\ 0, & r \neq s. \end{cases}$$

We also have

$$M_d^{r,s} = O\left(rq^{(1/p-1/2)(r+s)}\right).$$

Proof. By the Proposition 6.3 we have

$$M_d^{r,-s} = \sum_{\substack{m|(r,s) \\ mp|r-s \\ mp \nmid r}} \pi(m)m^2 + O\left(q^{(1/p-1/2)(r+s)}\right).$$

It is well known that $\pi(m) = q^m/m + O(q^{m/2}/m)$ (see [15, §2]). If $r = s$ then

$$\sum_{\substack{m|r \\ mp \nmid r}} \pi(m)m^2 = rq^r + O(rq^{r/2})$$

(note that except for $m = 2$ and possibly $m = r/2$ the other terms are negligible), which implies the assertion of the theorem for the case $r = s$. If $r > s$ then $(r, s) \leq s/2$ and

$$\sum_{\substack{m|(r,s) \\ mp|r-s \\ mp \nmid r}} \pi(m)m^2 = O(rq^{s/2}),$$

which implies the assertion of the theorem for $r \neq s$. The second part of the Theorem follows similarly from the second part of Proposition 6.3 (note that this time only m s.t. $mp|r + s$ contribute to the sum). \square

The last result should be compared with the following (see [4, Thm. 2]):

$$\langle T_U^r \overline{T_U^s} \rangle_{U \in \mathbf{U}_N} = \delta_{r,s} \min(r, N),$$

$$\langle T_U^r T_U^s \rangle_{U \in \mathbf{U}_N} = 0$$

(for $r, s > 0$).

6.2 Proof of Theorem 3

For simplicity we will prove Theorem 3 for $\theta = 0$, the proof of the general case proceeds with only slight modifications. We will write $S^2(f, \psi)$ instead of $S_0^2(f, \psi)$ for the 2-level density function.

Lemma 6.4.

$$S^2(f, \psi) = \frac{1}{(d-1)^2} \sum_{r,s=-\infty}^{\infty} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right) (T_{f,\psi}^{r,s} - T_{f,\psi}^{r+s}).$$

Proof. We have

$$S^2(f, \psi) = \sum_{j,k=1}^{d-1} v_{d-1}(\theta_j, \theta_k) - \sum_{j=1}^{d-1} v_{d-1}(\theta_j, \theta_j). \quad (19)$$

The Fourier series coefficients of the bi-periodic function v_{d-1} are given by

$$\begin{aligned} \hat{v}_{d-1}(r, s) &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} v_{d-1}(t, u) e^{-irt - isu} dt du = \\ &= \frac{1}{(d-1)^2} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right), \end{aligned}$$

the derivation is by a standard calculation, similar to that of 12. Since v_{d-1} is smooth the following holds for $|z| = |w| = 1$:

$$\begin{aligned} v_{d-1}(\arg z, \arg w) &= \sum_{r,s=-\infty}^{\infty} \hat{v}_{d-1}(r, s) z^r w^s = \\ &= \frac{1}{(d-1)^2} \sum_{r,s=-\infty}^{\infty} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right) z^r w^s. \end{aligned}$$

Now if $\rho_1, \dots, \rho_{d-1}$ are the normalised zeroes of $L_{f,\psi}$ and $\rho_j = e^{i\theta_j}$ then

$$\begin{aligned} \sum_{j,k=1}^{d-1} v_{d-1}(\theta_j, \theta_k) &= \frac{1}{(d-1)^2} \sum_{j,k=1}^{d-1} \sum_{r,s=-\infty}^{\infty} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right) \rho_j^r \rho_k^s = \\ &= \frac{1}{(d-1)^2} \sum_{r,s=-\infty}^{\infty} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right) T_{f,\psi}^{r,s}, \quad (20) \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{d-1} v_{d-1}(\theta_j, \theta_j) &= \frac{1}{(d-1)^2} \sum_{j=1}^{d-1} \sum_{r,s=-\infty}^{\infty} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right) \rho_j^{r+s} = \\ &= \frac{1}{(d-1)^2} \sum_{r,s=-\infty}^{\infty} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right) T_{f,\psi}^{r+s}. \quad (21) \end{aligned}$$

Combining (19), (20) and (21) we obtain the statement of the Lemma. \square

Corollary 6.5.

$$\langle S^2(f, \psi) \rangle_{f \in \mathcal{F}_d} = \frac{1}{(d-1)^2} \sum_{r,s=1}^{\infty} \hat{V} \left(\frac{r}{d-1}, \frac{s}{d-1} \right) (M_d^{r,s} - M_d^{r+s}).$$

Proof. Just average the previous lemma over \mathcal{F}_d . \square

We need one more lemma:

Lemma 6.6. *Let $V \in \mathcal{S}(\mathbf{R}^2)$ be a window function, \hat{V} its Fourier transform. Denote $K(\sigma) = \max(1 - |\sigma|, 0)$. We have*

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{V}(\sigma, -\sigma) K(\sigma) d\sigma &= \hat{V}(0, 0) - \\ &- \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(t, u) \left(1 - \left(\frac{\sin((t-u)/2)}{(t-u)/2} \right)^2 \right) dt du. \end{aligned}$$

Proof. Define $Y(\tau) = \int_{-\infty}^{\infty} V(t + \tau, t) dt$. We have $Y \in \mathcal{S}(\mathbf{R})$. It is easy to see from the definitions and Fubini's theorem that the Fourier transform of $Y(\tau)$ is $\hat{Y}(\sigma) = 2\pi \hat{V}(\sigma, -\sigma)$. The Fourier transform of the function $(\sin(\tau/2)/(\tau/2))^2$ is $K(\sigma)$, so we have by Plancherel's theorem

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{V}(\sigma, -\sigma) K(\sigma) d\sigma &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{Y}(\sigma) K(\sigma) d\sigma \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} Y(\tau) \left(\frac{\sin(\tau/2)}{\tau/2} \right)^2 d\tau = \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(t, u) \left(\frac{\sin((t-u)/2)}{(t-u)/2} \right)^2 dt du = \\ &= \hat{V}(0, 0) - \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(t, u) \left(1 - \left(\frac{\sin((t-u)/2)}{(t-u)/2} \right)^2 \right) dt du. \end{aligned}$$

\square

Now we are ready to prove Theorem 3. Assume $p > 2$. Let $V \in \mathcal{S}(\mathbf{R}^2)$ be s.t. \hat{V} is supported on $|\eta| + |\xi| \leq 1$. By Corollary 6.5 we have

$$\langle S^2(f, \psi) \rangle_{f \in \mathcal{F}_d} = \frac{1}{(d-1)^2} \sum_{\substack{r,s \in \mathbf{Z} \\ |r|+|s| < d}} \hat{V} \left(\frac{r}{d-1}, \frac{s}{d-1} \right) (M_d^{r,s} - M_d^{r+s}). \quad (22)$$

First we bound the contribution to the sum (22) of r, s s.t. $r \neq -s$. By Theorem 1 we have

$$\sum_{\substack{r,s \in \mathbf{Z} \\ |r|+|s| < d \\ r \neq -s}} M_d^{r+s} = O \left(d \sum_{r=1}^d M_d^r \right) = O(d)$$

(note that $M_d^r = M_d^{-r}$), since M_d^r decreases geometrically in r for $0 < r < d$. Similarly, by Theorem 9 we also have

$$\sum_{\substack{r,s \in \mathbf{Z} \\ |r|+|s| < d \\ r \neq -s}} M_d^{r,s} = O(d)$$

as $M_d^{r,s}$ for $r \neq s$ decreases geometrically in $|r| + |s|$. The overall contribution to the RHS of (22) is $O(1/d)$ (note that \hat{V} is bounded).

It remains to estimate

$$\begin{aligned} \frac{1}{(d-1)^2} \sum_{-d \leq r \leq d} \hat{V}\left(\frac{r}{d-1}, \frac{s}{d-1}\right) (M_d^{r,-r} - M_d^0) = \\ = \frac{d}{d-1} \hat{V}(0,0) + \frac{1}{d-1} \sum_{\substack{-d/2 < r < d/2 \\ r \neq 0}} \hat{V}\left(\frac{r}{d-1}, \frac{-r}{d-1}\right) (M_d^{r,-r} - d+1) \end{aligned}$$

(note that $M_d^0 = d-1$, $M_d^{0,0} = (d-1)^2$). Invoking Theorem 9 and noting that the error terms accumulate to at most $O(1/d^2)$ (the error term for $M_d^{r,-r}$ in Theorem 9 decreases geometrically in r) we see that

$$\begin{aligned} \langle S^2(f, \psi) \rangle_{f \in \mathcal{F}_d} = \\ = \hat{V}(0,0) + \frac{1}{(d-1)^2} \sum_{\substack{-d/2 < r < d/2 \\ r \neq 0}} \hat{V}\left(\frac{r}{d-1}, \frac{-r}{d-1}\right) (|r| - d + 1) + O(1/d) = \\ = \hat{V}(0,0) + \sum_{\substack{-d/2 < r < d/2 \\ r \neq 0}} \left(\frac{|r|}{d-1} - 1\right) \hat{V}\left(\frac{r}{d-1}, \frac{-r}{d-1}\right) \frac{1}{d-1} + O(1/d) \rightarrow \\ \rightarrow \hat{V}(0,0) + \int_{-\infty}^{\infty} \hat{V}(\sigma, -\sigma) (|\sigma| - 1) d\sigma = \hat{V}(0,0) - \int_{-\infty}^{\infty} \hat{V}(\sigma, -\sigma) K(\sigma) d\sigma \end{aligned}$$

as $d \rightarrow \infty$ by the definition of the Riemann integral (we used the fact that $\hat{V}(\sigma, -\sigma)$ is supported on $[-1/2, 1/2]$). Now using Lemma 6.6 we obtain the assertion of Theorem 3.

7 Reformulation in terms of Dirichlet L-functions and generalisation

In the present section we will see that the family of L-functions $L_{f,\psi}$, $f \in \mathcal{F}_d$ is actually a special case of a family of Dirichlet L-functions corresponding to multiplicative characters of $\mathbf{F}_q[x]$ and generalise our main results to such families.

7.1 Dirichlet characters and L-functions

We briefly recall the basic properties of Dirichlet characters and L-functions over $\mathbf{F}_q[x]$. For details see [15, §4]. Let $Q(x) \in \mathbf{F}_q[x]$ be a monic polynomial of degree m and let $\chi : (\mathbf{F}_q[x]/Q)^\times \rightarrow \mathbf{C}^\times$ be a character of the multiplicative group of residues modulo Q . We may extend χ to $\mathbf{F}_q[x]$ by

$$\chi(g) = \begin{cases} \chi(g \bmod Q), & (g, Q) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The map $\chi : \mathbf{F}_q[x] \rightarrow \mathbf{C}$ thus defined is called a Dirichlet character. The character χ is called primitive if there is no proper divisor Q_1 of Q s.t. $\chi(g)$ for g prime to Q only depends on $g \bmod Q_1$. It is called trivial if it takes the value 1 on all polynomials prime to Q . It is called even if it takes the value 1 on constants and odd otherwise. We denote

$$e(\chi) = \begin{cases} 1, & \chi \text{ is even,} \\ 0, & \chi \text{ is odd.} \end{cases}$$

Denote by \mathcal{M} the set of monic polynomials in $\mathbf{F}_q[x]$ and by \mathcal{P} the set of monic irreducible polynomials in $\mathbf{F}_q[x]$. The L-function corresponding to the character χ is defined as follows:

$$L_\chi(z) = \sum_{g \in \mathcal{M}} \chi(g) z^{\deg g} = \prod_{h \in \mathcal{P}} (1 - \chi(h)z)^{-1}. \quad (23)$$

It turns out that for a primitive character χ modulo Q the function $L_\chi(z)$ is a polynomial of degree $d - 2$ if χ is even and $d - 1$ if χ is odd. Further it factors as follows:

$$L_\chi(z) = (1 - z)^{e(\chi)} \prod_{i=1}^{d-1-e(\chi)} (1 - \rho_i q^{1/2} z)$$

with $|\rho_i| = 1$. The ρ_i are called the normalised zeroes of $L_\chi(z)$.

7.2 Dirichlet characters corresponding to A-S curves

Let d be a natural number and $\psi \in \Psi$ a nontrivial additive character of \mathbf{F}_p . Let $f = \sum_{i=0}^d a_i x^i \in \mathcal{F}_d$ be a polynomial. We will attach a Dirichlet character χ modulo x^{d+1} to f, ψ . Let $g = \sum_{i=1}^k b_i x^i \in \mathbf{F}_q[x]$ be a polynomial. If $x|g$ we define $\chi(g) = 0$. Otherwise we may write

$$g(x) = \prod_{i=1}^k (1 - \alpha_i x), \quad (24)$$

where α_i are the roots of g in the algebraic closure of \mathbf{F}_q .

First we observe that for every natural j the quantity $\sum_{i=1}^k \alpha_i^j$ lies in \mathbf{F}_q and depends only on the coefficients b_0, \dots, b_j (in other words it depends only

on $g \bmod x^{d+1}$). As in section 5.1 we denote by σ_j the order j elementary symmetric function in the variables x_1, \dots, x_k . By (24) we have $\sigma_j(\alpha_1, \dots, \alpha_k) = (-1)^j b_j/b_0$. Using Newton's identity 10 recursively we can show that $\sum_{i=1}^k x_i^j$ can be expressed as a polynomial in $\sigma_l, 1 \leq l \leq \min(j, k)$ with integer coefficients. Therefore $\sum_{i=1}^k \alpha_i^j$ can be expressed as a polynomial in $b_1/b_0, \dots, b_j/b_0$ with integer coefficients and so it must lie in \mathbf{F}_q and depends only on $b_1/b_0, \dots, b_j/b_0$ (and therefore only on $g \bmod x^{d+1}$).

It follows from the above that the quantity

$$\sum_{i=1}^k f(\alpha_i) = \sum_{j=0}^d a_j \sum_{i=1}^k \alpha_i^j$$

is in \mathbf{F}_q and depends only on $b_1/b_0, \dots, b_{\min(d,k)}/b_0$. Now we define

$$\chi(g) = \psi \left(\mathfrak{t}_{q/p} \sum_{i=1}^k f(\alpha_i) \right). \quad (25)$$

By what we have seen $\chi(g)$ is well defined and depends only on $g \bmod x^{d+1}$. Further it is multiplicative, because the set of zeroes (counting multiplicity) of a product of two polynomials is the union of their sets of zeroes. Therefore χ is a Dirichlet character modulo x^{d+1} . Obviously for constant g we have $\chi(g) = 1$, so χ is even. By 25 we also have that χ^p is trivial, so χ is an order p character modulo x^{d+1} .

We fix $\psi \in \Psi$ and denote by χ_f the character constructed above for a given $f \in \mathcal{F}_d$.

Lemma 7.1. *For any $f \in \mathcal{F}_d$ the character χ_f is primitive. The characters $\chi_f, f \in \mathcal{F}_d$ are all distinct and any primitive character χ modulo x^{d+1} with χ^p trivial is of the form $\chi = \chi_f$ for some $f \in \mathcal{F}_d$.*

Proof. First we show that for $f \in \mathcal{F}_d$ the character $\chi = \chi_f$ is primitive. For this it is enough to show that for some $c \in \mathbf{F}_q$ we have $\chi(1 - cx^d) \neq 1$. Write $f = \sum_{i=0}^d a_i x^i$. It is easy to see from the definition that $\chi(1 - cx^d) = \psi(\mathfrak{t}_{q/p}(cda_d))$. Since $(d, p) = 1$ and $a_d \neq 0$ there exists $c \in \mathbf{F}_q$ s.t. $\mathfrak{t}_{q/p}(cda_d) \neq 0$.

Now let $f_1, f_2 \in \mathcal{F}_d$ s.t. $f_1 \neq f_2$. We will show that $\chi_{f_1} \neq \chi_{f_2}$. By the definition of \mathcal{F}_d there exists some $e \leq d$ s.t. $f = f_1 - f_2 \in \mathcal{F}_e$. Let $g \in \mathbf{F}_q[x]$ be some polynomial prime to x . Write it as $g(x) = g(0) \prod_{i=1}^k (1 - \alpha_i x)$ with $\alpha_i \in \mathbf{F}_{q^r}$ for some r . We have

$$\chi_{f_1}(g) \chi_{f_2}(g)^{-1} = \psi \left(\mathfrak{t}_{q/p} \sum_{i=1}^k f(\alpha) \right).$$

It is enough to show that for some g we have $\mathfrak{t}_{q/p} \sum_{i=1}^k f(\alpha) \neq 0$. Taking $g = 1 - cx^e$ we have (as above) $\chi(g) = \psi(\mathfrak{t}_{q/p}(cea_e))$ where $f = \sum_{i=0}^e a_i x^i$ and using the fact that $(e, p) = 1$ and $a_e \neq 0$ we can pick c so that $\chi(g) \neq 1$.

We have seen that the correspondence $f \mapsto \chi_f$ is one-to-one from \mathcal{F}_d to the set of order p primitive characters modulo x^{d+1} . To show that it is onto it is enough to show that these sets are identical in size. Recall that $\#\mathcal{F}_d = (q-1)q^{d-\lfloor d/p \rfloor - 1}$. For a finite abelian group G denote by G^* its dual and by $G[m]$ the m -torsion of the group G . The groups G, G^* are always isomorphic. Take $G = (\mathbf{F}_q[x]/x^{d+1})^\times$. First we compute $\#G^*[p] = \#G[p]$, which is the number of all order p characters modulo x^{d+1} . Each element of G is represented uniquely by a polynomial $g(x) = \sum_{i=0}^d c_i x^i$. We have $g(x)^p = \sum_{i=1}^d c_i x^{pi}$ and $g(x)^p \equiv 1 \pmod{x^{d+1}}$ iff $c_1 = c_2 = \dots = c_{\lfloor d/p \rfloor} = 0$ and $c_0 = 1$. Therefore $\#G^*[p] = \#G[p] = q^{d-\lfloor d/p \rfloor + 1}$ and this is also the number of order p characters modulo x^{d+1} . By the same reasoning the number of order p characters modulo x^d is $q^{d-\lfloor (d-1)/p \rfloor} = q^{d-\lfloor d/p \rfloor}$ (since $(d, p) = 1$) and so the number of primitive characters modulo x^{d+1} equals $q^{d-\lfloor d/p \rfloor + 1} - q^{d-\lfloor d/p \rfloor} = (q-1)q^{d-\lfloor d/p \rfloor - 1} = \#\mathcal{F}_d$. \square

Lemma 7.2. *For the character χ defined above we have $L_\chi(z) = (1-z)L_{f,\psi}$. In particular the normalised nontrivial zeroes of $L_\chi(z)$ coincide with the normalised zeroes of $L_{f,\psi}$.*

Proof. Since both $L_\chi(z)$ and $(1-z)L_{f,\psi}$ have constant coefficient 1 it is enough to show that

$$\frac{d}{dz} \log L_\chi(z) = \frac{d}{dz} \log ((1-z)L_{f,\psi}).$$

Let r be a natural number, $\alpha \in \mathbf{F}_{q^r}$ an element with minimal polynomial h (over \mathbf{F}_q) of degree $s|r$. Denote by $\alpha_1 = \alpha, \dots, \alpha_s$ the roots of h in \mathbf{F}_{q^r} . We have $t_{q^r/p}(f(\alpha)) = \sum_{i=1}^r f(\alpha^{p^i}) = \frac{r}{s} \sum_{i=1}^s f(\alpha_i)$ and so for $\alpha \neq 0$ we have $\psi(t_{q^r/p}(f(\alpha))) = \chi(h^*)^{r/s}$ (recall that $h^* = \sum_{i=0}^s c_{s-i} x^i$ where $h = \sum_{i=0}^s x^i$ and it has roots $\alpha_1^{-1}, \dots, \alpha_s^{-1}$). For $\alpha = 0$ we have $t_{q^r/p}(f(\alpha)) = rf(0)$.

Using the above and 23 we obtain

$$\begin{aligned} \frac{d}{dz} \log L_{f,\psi} &= \sum_{r=1}^{\infty} \sum_{\alpha \in \mathbf{F}_{q^r}} \psi(t_{q^r/p} f(\alpha)) z^{r-1} = \\ &= \frac{1}{z} \sum_{r=1}^{\infty} \sum_{h \in \mathcal{P}, \deg h | r, h \neq x} \deg(h) \chi(h^*)^{r/\deg(h)} z^r + \frac{1}{z} \sum_{r=1}^{\infty} \psi(t_{q/p} f(0))^r z^r = \\ &= \frac{1}{z} \sum_{h \in \mathcal{P}} \sum_{k=1}^{\infty} \deg(h) \chi(h)^k z^{k \deg h} + \frac{1}{1-z} = \sum_{h \in \mathcal{P}} \frac{\deg(h) \chi(h) z^{\deg h - 1}}{1 - \chi(h) z^{\deg h}} + \frac{1}{1-z} = \\ &= \frac{d}{dz} \log \left(\prod_{h \in \mathcal{P}} (1 - \chi(h) z^{\deg h})^{-1} \right) - \frac{d}{dz} \log(1-z) = \frac{d}{dz} \log (L_{\chi_f} (1-z)^{-1}) \end{aligned}$$

(we used the fact that the operation $h \mapsto h^*$ permutes $\mathcal{P} \setminus \{x\}$, that $\chi(x) = 0$ and that $f(0) = 0$ for $f \in \mathcal{F}_d$). \square

We see that the family of L-functions $L_{f,\psi}(z), f \in \mathcal{F}_d$ coincides with the family of L-functions $L_\chi^*(z) = (1-z)^{-1} L_\chi(z)$ where χ ranges over the order

p primitive characters modulo x^{d+1} . Next we generalise Theorems 1 and 2 to more general families of Dirichlet characters, obtaining a new (but essentially equivalent) proof of our results for A-S L-functions.

7.3 The family of Dirichlet L-functions corresponding to a subgroup of $(\mathbf{F}_q[x]/Q)^\times$

For any finite Abelian group A we denote by A^* its dual group. Let $Q(x) \in \mathbf{F}_q[x]$ be a monic polynomial of degree m . Denote by G the group of characters modulo Q , which we will also identify with $(\mathbf{F}_q[x]/Q)^{\times*}$, i.e. we view the elements of G also as characters of $(\mathbf{F}_q[x]/Q)^\times$. Let H be a subgroup of G . For any $Q'|Q$ we denote by $H_{Q'}$ the subgroup of H consisting of the characters which have period Q' (or a divisor of Q'). Denote by H' the set of primitive characters in H . Denote by H^\perp the subgroup of $(\mathbf{F}_q[x]/Q)^\times$ consisting of elements g s.t. $\chi(g) = 1$ for all $\chi \in H$. It is the subgroup of $(\mathbf{F}_q[x]/Q)^\times$ orthogonal to $H \subset (\mathbf{F}_q[x]/Q)^{\times*}$ and its order is $\#H^\perp = \#G/\#H$ (this relation holds for any finite abelian group). The following orthogonality relation holds for $g \in \mathbf{F}_q[x]$:

$$\sum_{\chi \in H} \chi(g) = \begin{cases} \#G/\#H, & g \bmod Q \in H^\perp \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

We denote by $\mathcal{P}(H)$ the set of monic irreducible polynomials h s.t. $h \bmod Q \in H^\perp$.

Let χ be an even primitive character modulo Q . Recall that its L-function can be factored as

$$L_\chi(z) = (1-z) \prod_{i=1}^{m-2} (1 - q^{1/2} \rho_i), \quad (27)$$

with $|\rho_i| = 1$ (ρ_i are the normalised zeroes of the L-function).

Lemma 7.3. *Let r be a natural number.*

$$\sum_{i=1}^{m-2} \rho_i^r = -q^{-r/2} - q^{-r/2} \sum_{h \in \mathcal{P}, \deg h | r} (\deg h) \chi(h)^{r/\deg h}.$$

Proof. By 23 and 27 we have

$$\begin{aligned}
\sum_{r=1}^{\infty} q^{r/2} \left(q^{-r/2} + \sum_{i=1}^{m-2} \rho_i^r \right) z^r &= z \left(\frac{1}{1-z} + \sum_{i=1}^{m-2} \frac{q^{1/2} \rho_i}{1 - q^{1/2} \rho_i z} \right) = \\
&= -z \frac{d}{dz} \log \left((1-z) \prod_{i=1}^{m-2} (1 - q^{1/2} \rho_i z) \right) = -z \frac{d}{dz} \log L_{\chi}(z) = \\
&= - \sum_{h \in \mathcal{P}} \frac{\chi(h) (\deg h) z^{\deg h}}{1 - \chi(h) z^{\deg h}} = \sum_{r=1}^{\infty} \sum_{h \in \mathcal{P}} (\deg h) \chi(h)^r z^{r \deg h} = \\
&= \sum_{r=1}^{\infty} \left(\sum_{h \in \mathcal{P}, \deg h | r} (\deg h) \chi(h)^{r/\deg h} \right) z^r.
\end{aligned}$$

Comparing coefficients at z^r we obtain the statement of the lemma. \square

We denote $T_{\chi}^r = \sum_{i=1}^{m-2} \rho_i^r$. For a group of Dirichlet characters J modulo Q and a natural number s denote $\eta(J, s) = \#\{h \in \mathcal{P}(J) \mid \deg h = s\}$. For a nonzero polynomial $P \in \mathbf{F}_q[x]$ with factorisation $P = P_1 \dots P_k$ into irreducibles we denote

$$\mu(P) = \begin{cases} (-1)^k, & P \text{ is squarefree,} \\ 0, & \text{otherwise} \end{cases}$$

(this is the Möbius function on $\mathbf{F}_q[x]$). For an abelian group A and natural number k denote by A^k the subgroup of k -th powers in A . Finally denote by H^{pr} the set of primitive characters in H and $M_H^r = \langle T_{\chi}^r \rangle_{\chi \in H^{\text{pr}}}$. The following proposition is a generalisation of Proposition 5.6 to an arbitrary family of Dirichlet characters corresponding to a subgroup H of characters modulo Q .

Proposition 7.4.

$$M_H^r = \langle T_{\chi}^r \rangle_{\chi \in H^{\text{pr}}} = -q^{r/2} - \frac{q^{-r/2}}{\#H^{\text{pr}}} \sum_{Q'|Q} \mu\left(\frac{Q}{Q'}\right) \#H_{Q'} \sum_{s|r} s \cdot \eta\left(H_{Q'}^{r/s}, s\right),$$

where $\sum_{Q'|Q}$ denotes summation over monic divisors Q' of Q .

Proof. First of all it follows from the inclusion-exclusion principle that for any map $X : H \rightarrow \mathbf{C}$ we have

$$\sum_{\chi \in H^{\text{pr}}} X(\chi) = \sum_{Q'|Q} \mu(Q/Q') \sum_{\chi \in H_{Q'}} X(\chi). \quad (28)$$

Next, by Lemma 7.3 and the orthogonality relation (26) for any $Q'|Q$ we have

$$\begin{aligned}
\sum_{\chi \in H_{Q'}} (T_{\chi}^r + q^{r/2}) &= -q^{-r/2} \cdot \#H_{Q'} \cdot \sum_{\substack{h \in \mathcal{P} \\ \deg h | r}} \deg h = \\
&= -q^{-r/2} \cdot \#H_{Q'} \sum_{s|r} s \cdot \eta\left(H_{Q'}^{r/s}, s\right)
\end{aligned}$$

(we used the fact that $h^{r/s} \bmod Q' \in H_{Q'}^\perp$ iff $h \bmod Q' \in (H_{Q'}^{r/s})^\perp$). Combining this with (28) we obtain the statement of the proposition. \square

The last proposition can be used to obtain bounds on M_H^r . For example assume that Q is irreducible. Then it follows from the proposition that for $r \geq m$ we have $M_H^r = O(rq^{r/2-m} \#H^\perp)$, since the total number of monic polynomials h with $\deg h = r$, $h \bmod Q \in H^\perp$ is $q^{r-m} \#H^\perp$. For $r < m$ we have $M_H^r = O(rq^{-r/2} \#H^\perp)$.

For another example take $Q = x^{d+1}$ and $H = ((\mathbf{F}_q[x]/Q)^\times)^*[p]$ (the group of order p characters modulo Q). We have seen in section 7.2 that $H^{\text{pr}} = \{\chi_f\}_{f \in \mathcal{F}_d}$, so $M_H^r = M_d^r$. Proposition 5.6 now follows from proposition 7.4. Indeed the only divisors $Q'|Q$ s.t. $\mu(Q/Q') \neq 0$ are $Q' = x^{d+1}, x^d$ and $H_{Q'}^\perp$ consists of the (invertible) p -th powers modulo Q' , which are represented by polynomials of the form $a_0 + a_1 x^p + \dots + a_{\lfloor d'/p \rfloor} x^{\lfloor d'/p \rfloor p}$ where $d' = \deg Q' - 1$. Combining this with the fact that $H^k = H$ if $(k, p) = 1$ and H^k is trivial if $p|k$ the formula in proposition 7.4 translates into the one in proposition 5.6 (note that for $s < d$ we have $\eta(H^\perp, s) = 0$, since the classes in H^\perp are represented by p -th powers). Our main theorems follow from the latter proposition.

8 A-S family with odd polynomials

Throughout this section $p > 2$. Let d be odd and as usual $(d, p) = 1$. Denote by \mathcal{O}_d the subset of odd polynomials in \mathcal{F}_d , i.e. polynomials $f \in \mathcal{F}_d$ satisfying $f(-x) = -f(x)$, in other words only odd powers of x appear in f . We call \mathcal{O}_d as well as the corresponding family of curves and L-functions the odd A-S family. As with \mathcal{F}_d the family $\{L_{f,\psi}\}_{f \in \mathcal{O}_d}$ does not depend on the choice of $\psi \in \Psi$. In the present section we formulate conjectures for \mathcal{O}_d analogous to our main results for \mathcal{F}_d based on a random symplectic matrix model. However we will only be able to prove a very weak result in this direction.

Lemma 8.1. *For $f \in \mathcal{O}_d, \psi \in \Psi$ we have $L_f(z) \in \mathbf{R}[z]$.*

Proof. By (3) it would suffice to show that $\sum_{\alpha \in \mathbf{F}_{q^r}} \psi(t_{q^r/p} f(\alpha)) \in \mathbf{R}$ for every natural r . Since f is an odd polynomial we have $f(-\alpha) = -f(\alpha)$ for $\alpha \in \mathbf{F}_{q^r}$ and so $\psi(t_{q^r/p} f(-\alpha)) = \overline{\psi(t_{q^r/p} f(\alpha))}$ and partitioning $\mathbf{F}_{q^r} \setminus \{0\}$ into pairs $\alpha, -\alpha$ we obtain $\sum_{\alpha \in \mathbf{F}_{q^r}} \psi(t_{q^r/p} f(\alpha)) \in \mathbf{R}$. \square

The latter fact suggests that we model the set of normalised L-zeroes of a random $f \in \mathcal{O}_d$ by the set of eigenvalues of a random matrix $U \in \mathbf{USp}_{d-1}$ (we denote thus the unitary symplectic group), taken uniformly w.r.t. the Haar measure. Note that the characteristic polynomial of a unitary symplectic matrix has real coefficients. This is the model usually used for the L-zeroes of a family of curves over a finite field, provided that the corresponding L-functions do not split into primitive L-functions with non-real coefficients, as it happens for the entire A-S family if $p > 2$. A more compelling reason for considering the

random symplectic matrix model is an equidistribution result due to N. Katz and P. Sarnak for similar (and more general) families of L-functions with fixed d and $q \rightarrow \infty$, see Theorem 3.10.7 in [7].

For the unitary symplectic group the following holds:

$$\langle T_U^r \rangle_{U \in \mathbf{USp}_{d-1}} = \begin{cases} -e_{2,r}, & r < d, \\ 0, & r \geq d, \end{cases}$$

see [4, §4].

We conjecture the following

Conjecture 8.2. *There exists a constant $\delta > 0$ such that for any $\epsilon > 0$ we have*

$$\langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d} = \begin{cases} -e_{2,r}, & r < d \\ 0, & r \geq d \end{cases} + O_\epsilon(q^{\epsilon r - \delta d} + q^{-\delta r}).$$

From this one can derive using the method of section 5.2 the following

Conjecture 8.3. *Let $V \in \mathcal{S}(\mathbf{R})$ be a window function,*

$$v_d(t) = \sum_{n=-\infty}^{\infty} V(d(t + 2\pi n)).$$

For $f \in \mathcal{O}_d$ denote $Z_f = \sum_{j=1}^{d-1} v_d(\theta_j)$, where $\rho_j = e^{i\theta_j}$ are the normalised zeroes of $L_{f,\psi}$. Similarly for a matrix $U \in \mathbf{USp}_{d-1}$ with eigenvalues ρ_i denote $Z_U = \sum_{j=1}^{d-1} w_d(\theta_j)$. Then

$$\langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d} \rightarrow \langle T_U^r \rangle_{U \in \mathbf{USp}_{d-1}}$$

as $d \rightarrow \infty$.

A possible approach to estimating $\langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d}$ is to reformulate the problem in terms of a family of Dirichlet characters and use Proposition 7.4. We keep the notation of section 7.1. We take $Q = x^{d+1}$. Recall that to any polynomial $f \in \mathbf{F}_q[x]$ with $\deg f \leq d$ we can attach a character χ_f modulo x^{d+1} defined by (24), (25). It is primitive iff $\deg f = d$. We also have $\chi_{f_1 f_2} = \chi_{f_1} \chi_{f_2}$ if $\deg f_1, \deg f_2 \leq d$. Denote $G = ((\mathbf{F}_q[x]/Q)^\times)^*$. As usual we identify G with the group of characters modulo Q . Denote $H = \{\chi_f | f \in \mathbf{F}_q[x], \deg f \leq r, f(-x) = -f(x)\}$. By the above remarks this is a subgroup of G . We have $H^{\text{pr}} = \{\chi_f\}_{f \in \mathcal{O}_d}$. By Lemma 7.2 we have

$$\langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d} = \langle T_\chi^r \rangle_{\chi \in H^{\text{pr}}}.$$

The only monic divisor $Q'|Q$ s.t. $\mu(Q/Q') \neq 0$ is $Q' = x^d$ which we denote by Q_1 . Obviously for any natural $s|r$ we have $\eta(H^{r/s}, s) = O(q^s)$. Also we have $\#H/\#H^{\text{pr}} = q/(q-1)$, $\#H_{Q_1}/\#H^{\text{pr}} = 1/(q-1)$. Note also that $H^2 = H$ since H consists of order p characters. Proposition 7.4 now implies

Proposition 8.4.

$$\begin{aligned} \langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d} &= \\ &= -\frac{rq^{-r/2}}{q-1} \left(q\eta(H, r) + e_{2,r} \frac{q}{2} \eta(H, r/2) - \eta(H_{Q_1}, r) - e_{2,r} \frac{1}{2} \eta(H_{Q_1}, r/2) \right) + \\ &\quad + O\left(rq^{-r/6}\right). \end{aligned}$$

Lemma 8.5. *The group H^\perp consists of residues of the form*

$$g_1(x^p)g_2(x^2) \bmod Q$$

where $g_i \in \mathbf{F}_q[x]$, $(g_i, x) = 1$. The same holds for $H_{Q_1}^\perp$ and Q_1 respectively.

Proof. Take any $f \in \mathcal{O}_d$ and $g_1 \in \mathbf{F}_q[x]$, $(g_1, x) = 1$. Since χ_f is an order p character we have $\chi_f(g_2(x^p)) = \chi_f(g_1(x)^p) = 1$. Now take $g_2 \in \mathbf{F}_q[x]$, $(g_2, x) = 1$. We may assume $g_2 = 1 + b_1x + \dots + b_kx^k$ since χ_f is even. The inverse roots of $g_2(x)$ come in pairs $\pm\alpha$ and since f is odd by 25 we have $\chi_f(g_2(x^2)) = 1$. It remains to note that the group of residues of the form $g_1(x^p)g_2(x^2) \bmod Q$ has order $(q-1)q^{\lfloor d/p \rfloor + (d+1)/2 - \lfloor d/2p \rfloor}$ and so does H . The same argument works for H_{Q_1} . \square

Now the problem of estimating $\langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d}$ reduces to estimating the number of monic irreducible polynomials of degree r (and $r/2$ if r is even) which can be written in the form $h \equiv g_1(x^p)g_2(x^2) \pmod{x^d}$ for some $g_i \in \mathbf{F}_q[x]$, $(g_i, x) = 1$ (and the same for x^{d+1}). Conjecture 8.2 follows from heuristics about the number of irreducible polynomials of given degree falling in the subgroups $H^\perp, H_{Q_1}^\perp$ modulo x^d, x^{d+1} respectively.

The following conjecture, if proven, would settle the case $r < d/4$:

Conjecture 8.6. *Let q be a power of a prime $p > 2$, d a natural number. Let $h \in \mathbf{F}_q[x]$ be prime to x , $\deg h = r$ and $r < d/4$. Assume that there exist $g_1, g_2 \in \mathbf{F}_q[x]$ s.t. $h \equiv g_1(x^p)g_2(x^2) \pmod{x^d}$. Then for all sufficiently large d there in fact exist $g_3, g_4 \in \mathbf{F}_q[x]$ s.t. $h = g_3(x^p)g_4(x^2)$. In particular if h is irreducible then $h = g_4(x^2)$.*

We will give some evidence for Conjecture 8.6, namely we will show that it holds for $r < Cp \log_q d$ with any constant $0 < C < 1$. First we show how Conjecture 8.6 implies Conjecture 8.2 for $r < d/4$. First assume that r is odd. Then by Conjecture 8.6 and Lemma 8.5 we have $\eta(H, r) = \eta(H_{Q_1}, r) = 0$ and so by Proposition 8.4 we have $\langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d} = O(rq^{-r/6})$. Now assume that r is even. Consider first $\eta(H, r)$. By Conjecture 8.6 and Lemma 8.5 any irreducible polynomial of degree r the residue of which modulo Q is orthogonal to H is of the form $g(x^2)$ with $g \in \mathbf{F}_q[x]$ and $\deg g = r/2$. The polynomial $g(x^2)$ is irreducible iff g is irreducible and any root of g in $\mathbf{F}_{q^{r/2}}$ is not a square in $\mathbf{F}_{q^{r/2}}$. The number of such (monic) g is easily seen to be $\frac{q^{r/2}}{2r} + O(q^{r/4})$, so

$\eta(H, r) = \frac{q^{r/2}}{2r} + O(q^{r/4})$. We also see from Conjecture 8.6 that $\eta(H_{Q_1}, r) = \eta(H, r)$ and $\eta(H, r/2) = \eta(H_{Q_1}, r/2) = O(q^{r/4})$, so by Proposition 8.4 we obtain $\langle T_{f,\psi}^r \rangle_{f \in \mathcal{O}_d} = -e_{2,r} + O(rq^{-r/6})$.

We see that to establish Theorem 4 it suffices to prove Conjecture 8.6 in the case $r < Cp \log_q d$.

8.1 Proof of Conjecture 8.6 for $r < Cp \log_q d$

Let q be a power of a prime $p > 2$ and $r < d$ natural numbers. Let $h \in \mathbf{F}_q[x]$ be a polynomial prime to x with $\deg h = r$. Suppose that h can be written in the form $h \equiv g_1(x^p)g_2(x^2) \pmod{x^d}$. Since the polynomials of the form $g(x^p)$ modulo x^d (prime to x) form a group we may also write $g_1(x^p)h \equiv g_2(x^2) \pmod{x^d}$ (for a different choice of g_1). Write $g_1 = \sum_{i=0}^{\infty} a_i x^i$ (for sufficiently large i we have $a_i = 0$). We may assume $p \leq r$, otherwise it is easy to see that for $g_1(x^p)h$ to be of the form $g_2(x^2)$ modulo x^d the polynomial h itself must be even. Now take any $C < 1$ and assume that $r < Cp \log_q d$. For sufficiently large d we have $\lfloor d/2p \rfloor > q^{r/p+1} + r/p + 1$. By the pigeonhole principle for some $j < k < d/p$ we have $a_{j+i} = a_{k+i}$ for $0 \leq i \leq r/p$, with j, k having the same parity. Now denote by g_3 the infinite power series

$$g_3(x) = \sum_{i=0}^{k-1} a_i x^i + \sum_{l=0}^{\infty} \sum_{i=0}^{k-j-1} a_i x^{l(k-j)+j+i}.$$

The first $k + \lfloor 2r/p \rfloor$ coefficients of g_3 coincide with those of g_1 and then the sequence of coefficients continues periodically with period $k-j$. The coefficients of $g_3(x^p)h$ coincide with those of $g_1(x^p)h$ up to the pk -th coefficient, after which they continue periodically with period $(k-j)p$. This is because each coefficient depends on at most $r/p + 1$ consecutive coefficients of g_3 (and the coefficients of h). When we multiply h by $g_1(x^p)$ the first d coefficients are zero for odd powers and the same holds for the first pk coefficients of $g_3(x^p)h$, after which it continues to hold by periodicity (since the period $(k-j)p$ is even). Thus the power series $g_3(x^p)h$ can be written in the form $g_4(x^2)$ for some power series $g_4(x)$. But g_3 is periodic and so must be g_4 . We then have two rational functions $h_1, h_2 \in \mathbf{F}_q(x)$ the x -adic expansions of which are g_3, g_4 respectively and we must have $h = h_1(x)^p/h_2(x^2)$. Now by the unique factorisation property in $\mathbf{F}_q[x]$ we see that h can be written in this form $h = h_3(x)^p h_4(x^2)$ with $h_i \in \mathbf{F}_q[x]$.

9 The distribution of the number of points on curves in the A-S family

In this section we consider the distribution of the number of points on the curve C_f as f varies uniformly through the family \mathcal{G}_d of all degree d monic polynomials in $\mathbf{F}_q[x]$ and $d \rightarrow \infty$ and prove Theorems 5,6,7,8. Throughout this section r is a fixed natural number.

9.1 Preliminaries

The number of \mathbf{F}_{q^r} -rational points on $C(f)$ equals the number of solutions to $F(x, y) = 0$ over \mathbf{F}_{q^r} plus one. From the Hilbert 90 theorem or elementary linear algebra it follows that for $a \in \mathbf{F}_{q^r}$ the equation $y^p - y = a$ is solvable in \mathbf{F}_{q^r} iff $\text{tr} a = 0$, in which case it has exactly p solutions, here by tr we denote the trace map from \mathbf{F}_{q^r} to \mathbf{F}_p . Thus for a given $x \in \mathbf{F}_{q^r}$ the equation $F(x, y) = 0$ is solvable iff $\text{tr} f(x) = 0$ and in this case it has exactly p solutions (see section 3.2). We denote by $N_r(f)$ the number of solutions to $\text{tr} f(x) = 0$ in \mathbf{F}_{q^r} . It is enough to study the distribution of $N_r(f)$ (the number of points on the curve is then $pN_r(f) + 1$).

From now on we fix r . Let $h \in \mathbf{F}_q[x]$ be an irreducible polynomial of degree $e|r$. Its splitting field is the subfield $\mathbf{F}_{q^e} \subset \mathbf{F}_{q^r}$ which is isomorphic to $\mathbf{F}_q[x]/h$. If $a \in \mathbf{F}_{q^r}$ is a root of h and $f \in \mathbf{F}_q[x]$ then we denote $\text{tr}_h f = \text{tr} f(a)$ (it does not depend on the choice of the root a). The value of $\text{tr}_h f$ only depends on the residue $f \bmod h$. Denote

$$\xi_h(f) = \begin{cases} 1, & \text{tr}_h f = 0 \\ 0, & \text{tr}_h f \neq 0. \end{cases}$$

Thus we have

$$N_r(f) = \sum_{e|d} e \sum_{\deg h=e} \xi_h(f), \quad (29)$$

where the inner sum is over monic irreducible h (henceforth h will always denote an irreducible polynomial in $\mathbf{F}_q[x]$ and summation over h will be always understood in this sense).

If p divides r/e then $\text{tr}_h f = 0$ for all $f \in \mathbf{F}_q[x]$. Otherwise exactly $1/p$ of the residues modulo h satisfy $\text{tr}_h f = 0$, because $\text{tr} : \mathbf{F}_q[x]/h \rightarrow \mathbf{F}_p$ is a nonzero \mathbf{F}_p -linear map.

The following lemmata will be used for the proof of the theorems.

Lemma 9.1. *Let $h_1, \dots, h_k \in \mathbf{F}_q[x]$ be distinct monic irreducible polynomials, $u = \sum_{i=1}^k \deg h_i$. Suppose $d \geq u$. Let $f \in \mathcal{G}_d$ be chosen uniformly at random. Then the values $f \bmod h_1, \dots, f \bmod h_k$ are distributed uniformly in $\oplus_{i=1}^k \mathbf{F}_q[x]/h_i$.*

Proof. The values $f \bmod h_i$ depend only on the residue of f modulo $\prod_{i=1}^k h_i$. If $d \geq u$ then \mathcal{G}_d can be divided into q^{d-u} complete systems of residues modulo any polynomial of degree d . \square

Lemma 9.2. *Suppose that for each natural m we have two sequences of random variables $X_1^{(m)}, \dots, X_{l(m)}^{(m)}$ and $Y_1^{(m)}, \dots, Y_{l(m)}^{(m)}$ satisfying the following conditions:*

1. $X_i^{(m)}$ and $Y_i^{(m)}$ have the same distribution for each m, i .
2. $Y_1^{(m)}, \dots, Y_{l(m)}^{(m)}$ are independent for each m .

3. For any fixed k there exists C_k s.t. for $m > C_k$ the variables $X_{i_1}^{(m)}, \dots, X_{i_k}^{(m)}$ are independent for any i_1, \dots, i_k .

Denote $S_m = \sum_{i=1}^{l(m)} X_i^{(m)}, T_m = \sum_{i=1}^{l(m)} Y_i^{(m)}$. Assume that there are sequences of real numbers A_m, B_m s.t. the distribution of $A_m T_m + B_m$ weakly converges to a distribution D as $m \rightarrow \infty$. Assume further that D is uniquely determined by its moments. Then $A_m S_m + B_m$ converges in distribution to D .

Proof. It is enough to show that for each k the k -th moment of $A_m S_m + B_m$ equals the k -th moment of $A_m T_m + B_m$ for $m > C_k$. This follows immediately from the assumed properties, the definition of the k -th moment and the multiplicativity of expectation on independent variables. \square

9.2 Proof of the results

For monic irreducible $h \in \mathbf{F}_q[x]$ and $f \in \mathbf{F}_q[x]$ denote

$$N(f) = \sum_{e|r} e \sum_{\deg h=e} \xi_h(f).$$

Lemma 9.1 shows that for any distinct h_1, \dots, h_k the variables $\xi_{h_1}(f), \dots, \xi_{h_k}(f)$ (f chosen uniformly from \mathcal{G}_d) are independent for $d \geq \sum \deg h_k$. We have $\xi_h(f) \sim B(1/p)$ if $(r/\deg h, p) = 1$ and $\xi_h(f) \equiv 1$ otherwise (recall that by $B(t)$ we denote the Bernoulli random variable taking the value 1 with probability t and 0 with probability $1 - t$). If $p|r$ then $\xi_h \equiv 1$ iff $\deg h|(r/p)$ and we have

$$\sum_{\deg h|(r/p)} (\deg h) \xi_h(f) = q^{r/p}$$

for any f . Now taking all the monic irreducible h s.t. $\deg h|r$ and noting that the sum of their degrees is $\sum_{e|r} e \nu(q, e) = q^r$ we obtain Theorem 1.

Now denote by $h_{e,1}, \dots, h_{e,\nu(q,e)} \in \mathbf{F}_q[x]$ the sequence of all monic irreducible polynomials of degree e . Given a sequence $q(m) = p(m)^{n(m)}$ denote by

$$Y_{e,i}, 1 \leq i \leq l(m) = \nu(q(m), e)$$

a set of independent random variables with $Y_{e,i}^{(m)} \sim eB(1/p)$ if $(r/e, p) = 1$ and $Y_{e,i}^{(m)} \equiv e$ if $p|r$. Taking $X_{e,i}^{(m)} = e\xi_{h_{e,i}}(f)$, we see from Lemma 9.1 that the conditions of Lemma 9.2 are satisfied for $X_{e,i}^{(m)}, Y_{e,i}^{(m)}$. Thus to establish theorems 6,7,8 we only need to show that

$$\sum_{e|r} \sum_{i=1}^{\nu(q(m),e)} Y_{e,i}^{(m)}$$

converges in distribution to the limit stated in the theorems. For the rest of this section we omit m from the notation (it is implicit).

In the setting of Theorem 6 we get (for $p > r$) a sum of p independent random variables distributed as $B(1/p)$, which converges to the Poissonian distribution with mean 1. In the setting of Theorem 7 it is enough to consider the variables corresponding to the irreducible polynomials of degree r and $r/2$ (if the latter is integral), because the number of polynomials of degree $e|r$ and $e < r/2$ is $O(q^{r/3})$. If r is odd then there are $\nu(q, r) = q^r/r + O(q^{r/3})$ variables $Y_{r,i}$ distributed like $rB(1/p)$, with mean r/p and variance $r^2p(1 - 1/p)$. The conclusion now follows from the central limit theorem.

If r is even and $p > 2$ then there are

$$\nu(q, r) = (q^r - q^{r/2})/r + O(q^{r/3})$$

variables $Y_{r,i}$ with mean r/p and variance $r^2p^{-1}(1 - 1/p)$ and $\nu(q, r/2) = q^{r/2}/r + O(q^{1/4})$ variables $Y_{r/2,i}$ with mean $r/2p$ and variance $r^2p^{-1}(1 - 1/p)/4$, which together gives the same result as for odd r . For $p = 2$ and r even there are $(q^r - q^{r/2})/r + O(q^{r/3})$ variables $Y_{r,i}$ distributed as $rB(1/p)$ and $q^{r/2}/r + O(q^{1/4})$ variables $Y_{r/2,i} \equiv 1$ and again the conclusion follows from the central limit theorem.

In the setting of Theorem 8 we can see as above that for odd r the limit distribution of our sum is the same as for $\sum_{i=1}^{\nu(q,r)} Y_{r,i}$. This is a sum of $\nu(q, r) = q^r/r + O(q^{r/3})$ independent random variables with distribution $rB(1/p)$. We can group each p such variables into a single variable with some distribution $S(p)$. The distribution $S(p)$ has mean r , variance $r(1 - 1/p)$ and a bound on the third moment independent of p (since the variable itself is bounded). We have a sum of $q^r/(rp) \rightarrow \infty$ variables with distribution $S(p)$, so invoking the effective version of the central limit theorem (see [5, §XVI.5]) we obtain the conclusion of Theorem 8.

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